# Contingent Claims Valuation of Optional Calling Plan Contracts in Telephone Industry

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**ABSTRACT**

This paper presents a valuation methodology for optional calling plan contracts on free-phone calls in the telephone industry. Utilization of the model is not limited to valuation and consequent decision making for the subscribers; it provides a useful guideline for telephone companies in designing calling plans and assessing subscribers' behavior.

## I. Introduction

Generally, an optional calling plan contract (hereafter OCP contract) is constituent of a price discount on telephone calls incurred during a billing period under the condition that a customer pays the option charge (i.e., up-front fixed fee) before the start of a billing period. OCP contracts are applied to either ordinary or free-phone calls (800-number services in the United States). With regards to the free-phone call, the receiving party pays telephone charge once a caller makes a call to a designated "free-phone" number, whereas the telephone chargeis paid for by the caller in an ordinary call. The free-phone call has become an important marketing tool in many service industries who wish to solicit customer inquiries and orders.

The purpose of this paper is to develop a methodology in evaluating a fair option charge for the OCP contract on free-phone call charges by applying a contingent claims approach. This paper is organized as follows. In the next section, we investigate basic properties of a fair option charge in relation to existing option pricing researches. We then develop a valuation methodology that yields a fair option charge and extend that methodology to the three representative OCP contracts. Finally, Section 3 concludes the paper.

## II. Modeling OCP Contracts

### 2.1. Distribution and preference free restrictions on the option charge

To begin, let us consider a very simple OCP contract for free-phone calls over a billing period from 0 to $T$ ($0<T$). Under this contract, a standard unit price $p$ is applied up to a certain amount $k$ and no charge is levied on the additional calls that exceed the level $k$ in exchange for an option charge $oc$. It is called a 3-part simple OCP in the telecommunications field, since three different prices (option charge $oc$, standard price $p$, price 0) are applied according to the accumulated call usage level (0, from 0 to $k$, more than $k$).

If we denote the instantaneous call usage at time $t$ ($0 \leq t \leq T$) by $x_t$, then the instantaneous free-phone call charge is $q_t = p \times x_t$ and the call charge $A_T$ for a billing period from 0 to $T$ is

$$A_T = \int_0^T q_t \, dt \qquad (1)$$

Since there is no additional charge over the amount $K$ (discount point, $K = p \times k$), the final call charge would be $A_T$ if $A_T < K$ and $K$ otherwise; that is, Min[$A_T$, $K$] or $A_T -$Max[0, $A_T - K$]. Notice that the second part of the last equation is the same as the payoff of a call option with an exercise price $K$ written on the uncertain call charge. Thus, subscribing to the OCP contract is identical to buying a call option from the telephone company. The option charge of the OCP contract, therefore, must be the same as the value of the call option.

In a complete market, customers can duplicate the payoffs of OCPs by constructing a traded security (or a dynamic portfolio) that has the same risk characteristics (i.e., is perfectly correlated) as their free-phone charge. Under the assumption that there is no arbitrage in the competitive OCP market (i.e., private agents can offer OCPs without restriction), the arbitrage relationships for European call options hold and the value of the fair option charge can be obtained by the use of the standard option pricing argument.

### 2.2. Valuing fair option charge of the 3-part simple OCP contracts

We will consider the problem of valuing the fair option charge of a 3-part simple OCP contract for the free-phone call. The interest rate is assumed to be a constant r and the instantaneous free-phone call charge $q_t$ follows a geometric Brownian motion process generated by the following stochastic differential equation

$$dq_t = \alpha q_t + \theta q_t dZ \tag{2}$$

The constant $\alpha$ reflects expected growth or decline in the level of free-phone call charges. The constant $\theta$ determines the instantaneous volatility of the process. The term $dZ$ represents a standard Wiener process, with mean zero and variance $dt$. The free-phone call charge process is correlated with the market portfolio of Merton (1973); the correlation coefficient is $\rho$. We present here an analytical approximation as a solution. This involves calculating the first two moments of the probability distribution of the sum exactly and then assuming that the distribution of the sum is lognormal with the same first two moments.

Let us define

$$M_1 = \frac{e^{\alpha * T} - 1}{\alpha^*} \tag{3}$$

$$M_2 = \frac{2e^{(2\alpha^* + \theta^2)T}}{(\alpha^* + \theta^2)(2\alpha^* \theta^*)} \frac{2}{\alpha^*}\left[\frac{1}{(2\alpha^* + \theta^2)} - \frac{e^{\alpha^*}}{\alpha^* + \theta^2)}\right] \tag{4}$$

Then, the first and second moments of the sum as seen at time zero for a period of time $T$ are $qM_1$ and $qM_2$. If we define

$$\mu = 2\ln(qM_1) - \ln(q^2 M_2)/$$

and $\sigma^2 = \ln(q^2 M_2) - \ln(qM_1$ ,

then the analytical approximation of the fair option charge of an OCP contract can be expressed as follows;

$$O = e^{-rT}[e^{(\mu + \frac{\sigma^2}{2})} \cdot N(d_1) - K \cdot N(d \tag{5}$$

where,

$$d_1 = [-\ln(K) + \mu + c \quad ,$$

$$d_2 = d_1 -$$

N( · )=standard cumulative normal distribution.

From equation (5), we are able to conjecture that the option charge increases with the present call charge $q$, summing period $T$ and volatility $\theta$, since they offer more chance to exercise. But, contrary to the stock option, an increase in interest rates decreases the option charge. The option charge also decreases as the discount point increases.

### 2.3 Valuing the fair option charge of three representative OCP contracts

To this point, we have been exclusively concerned with a very simple OCP contract, which can be used as a building block for constructing more general OCP contracts. If there are no arbitrage opportunities, and if a payoff function of the OCP contract provided by telephone companies at the end of the billing period is equal to that of the portfolio constructed by 3 part simple OCPs, then the two fair option charges must be equal. This means that the valuation methodology for the 3-part simple OCP contract in the previous subsection could be used for the valuation of the more general OCP contracts which can be found in a telephone market. Here, we extend the methodology in Subsection 2.2 to three popular variants of the OCPs.

**The Block-of-Time Type OCP Contracts**. As shown in Section I, the call charge paid by the subscriber of the BOT type OCP contract is $S^{BOT} = \text{Max}[0, (1-p)(A_T-B)]$. Since the call charge of the nonsubscriber is $A_T$, the total amount of discounts that the contract subscriber is entitled to is;

$$D^B = A_T - (1-\pi)\cdot Max[A_T - B,\ 0] \quad (6)$$

This payoff from the contract, $D^B$, can be exactly duplicated by an appropriately chosen portfolio of 3-part simple OCP contracts. It is equal to the payoff from a portfolio composed of one unit of 3-part simple OCP contracts with a discount point of zero and $(\pi-1)$ units of contracts with a discount point of $B$. As a result, the fair option charge of BOT type OCPs can be expressed as;

$$O^B = O(q, A, 0; 0, T) + (\pi - 1)\cdot O(q, A, 0; B, T) \quad (7)$$

**The Volume-Discount-Program Type OCP Contracts**. The OCP contract that is most frequently used for the free-phone call is the VDP type OCP. This contract cuts off the price with a different discount rate for each range of call charges under thecondition that the customer pays the option charge. In other words, under the VDP type OCP contract, the discount rate $\pi_i$ is applied to the free-phone call charge between $K_i$ and $K_{i+1}$ (where, $K_{i+1}>K_i$, $i=0,1,2,3,\cdots,n$, $n<\infty$) and $\pi_n$ is applied to all charges above $K_n$. Hence, a telephone company's reimbursement function of the OCP contract is expressed by;

$$D^V = \pi_0\cdot A_T + (\pi_1-\pi_0)Max[A_T-K_1, 0] + \ldots + (\pi_n - \pi_{n-1})Max[A_T-K_n, 0] \quad (8)$$

Equation (8) is exactly duplicated by 3-part simple OCPs which comprise units of OCPs that have discount point Ki. Hence, the fair option charge of the VDP type OCP contract $O^V$ is

$$O^V = \pi_0 O(q, A, 0; 0, T) + (\pi_1-\pi_0)O(q, A, 0; K_1, T) + \ldots + (\pi_n-\pi_{n-1})O(q, A, 0; K_n, T) \quad (9)$$

Equation (9) indicates that the BOT type OCP contracts can be considered as a special case of VDP type OCPs with $\pi_0=1$, $\pi_1=\pi$, and $K_0=0$, $K_1=B$.

**The Minimum-Usage-Guarantee Type OCP Contracts**. The MUG type OCP contract does not levy an option charge to a customer; rather, it requires a customer's guarantee that his call charge during a billing period will not be less than $G$. When a customer violates his promise, he should pay a predetermined penalty $P$ at the end of the billing period. Here, we deal with the problem of determining the fair penalty $P$ when the customer guarantees that his free-phone call charge will be more than $G$, and the discount structure ( $\pi_i$, $K_i$, $i = 0,1,2,3,\cdots, n$) is determined.

The MUG type OCP contract can be regarded as a portfolio constituent by buying a VDP type OCP contract and selling a continuously sampled, full period, unweighted arithmetic average rate Asian cash-or-nothing put with an exercise price $G/T$ and an underlying asset $A_T$. If there is no arbitrage in the competitive OCP contract market, the cost of constructing this portfolio has to be zero.

The cash-or-nothing put would be like selling a 3-part simple OCP except that, although the holder receives the penalty amount, he is under no obligation to give the underlying asset. From equation (5), such a partial put option, $CP$, would be worth

$$CP = e^{-rT} \cdot P \cdot N(-d_2) \qquad (10)$$

where, $d_2 = [-\ln(G) + \mu]/\sigma$

to the holder. Therefore, the fair penalty $P$ for a MUG type OCP contract is given by

$$P = \frac{O^V}{e^{-rT} \cdot [N(-d_2)]} \qquad (11)$$

We have evaluated the fair option charges of three representative types of OCP contracts which we developed in Subsection 2.2. The methodology also can be used to determine the fair option charge of many other types of OCP contracts.

## IV. Conclusions

The application of the option pricing model to real-life problems is currently receiving increased attention, and the model has been applied in a variety of contexts. We applied the option pricing methodology to the valuation problem of OCP contracts in the telephone industry. We show that the payoff of a 3-part simple OCP contract is identical to that of a call option, and option pricing theory can provide useful guidelines for OCP designers. Using the 3-part simple OCP contracts as a building block, we developed a valuation model for the fair option charge of general OCPs found in the actual market. When customers choose a tariff among OCPs, since regulatory bodies prohibit providing custom tailored OCPs, the valuation model can be used as a tool in making subscription decisions by potential customers and in estimating the subscription demand by a telephone company. We also showed that the standard theoretical approach consistently undervalues the OCP contract. The valuation results explored in this paper could be extended to ordinary call OCP contracts as well as to optional pricing schemes in other industries, such as gas or electricity, where the customer's price elasticity is very small.

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