CONTROL PROBLEMS IN FUZZY SYSTEMS

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1. Introduction

Since Bellman and Zadeh [1] studied fuzzy systems, Nazaroff introduced the notion of a fuzzy topological polysystem and attempted to develop several conclusions about such systems. His work was the first effort toward a fuzzification of the topological aspects of the optimal control of dynamical polysystems, as contributed by Halkin [3].

He obtained an abstraction and formalization of the previous work on fuzzy systems in [1], accomplished by considering the concept of a fuzzy topological polysystem. Equivalently, his work can be considered as a fuzzification of results obtained by Bushaw [2] and Halkin [3,4]. The order theoretic structure of fuzzy topological polysystems was based primarily on that given by Halkin with an appeal to the topological machinery. Moreover there was an attempt to use the fuzzy relation concept in order to construct a state equation for fuzzy systems [7,8].

The purpose of this paper is to obtain a complete fuzzification of the topological aspects of the optimal control of dynamical polysystems as generalizations of results in [6,10]. Particularly, for a fuzzy topology we will adopt the notion of fuzzy topology defined by R.Lowen [5] and moreover we will consider a fuzzy relation to express the degree of the relationship: an event follows a given event Using these notions, we formalize fuzzy topological polysystems and discuss about an optimal solution for a fuzzy optimal control problems depending on the level of crispness.

2. Fuzzy topological spaces

Let X be a set. A *fuzzy set* A in X is a map $\mu_A : X \to [0,1]$, called an associated membership function. We denote $\operatorname{Supp} A = \{x \in X : \mu_A(x) > 0\}$ and $A_\alpha = \{x \in X : \mu_A(x) \geq \alpha\}$, the *$\alpha$-cut* of A for $\alpha \in [0,1]$. Let $F(X) =$ the collection of all fuzzy sets in X.

Definition 2.1. Let A and B be fuzzy sets in X with $\mu_A(x)$ and $\mu_B(x)$ their associated membership functions. In addition, let A^C denote the *complement of* A and the symbols $\vee$ and $\wedge$ denote the *join* (supremum) and *meet* (infimum), respectively, in $[0,1]$. Then for all $x \in X$,

$$A = B \Leftrightarrow \mu_A(x) = \mu_B(x),$$

$$A \subset B \Leftrightarrow \mu_A(x) \leq \mu_B(x),$$

$$A \cup B \Leftrightarrow \mu_{A\cup B}(x) = \mu_A(x) \vee \mu_B(x),$$

$$A \cap B \Leftrightarrow \mu_{A\cap B}(x) = \mu_A(x) \wedge \mu_B(x),$$

$$A^C \Leftrightarrow \mu_{A^C}(x) = 1 - \mu_A(x).$$

Definition 2.2. If I is an index set and if for each $i \in I$, A_i is a fuzzy set in X, then for all $x \in X$

$$C = \bigcup_I A_i \Leftrightarrow \mu_C(x) = \bigvee_I \{\mu_{A_i}(x)\},$$

$$D = \bigcap_I A_i \Leftrightarrow \mu_D(x) = \bigwedge_I \{\mu_{A_i}(x)\}.$$

Clearly, the membership functions for the empty fuzzy set $\emptyset$ and the universal fuzzy set X are $\mu_\emptyset(x) = 0$ and $\mu_X(x) = 1$, respectively. If $\mu_A(x) \in \{0,1\}$ for all $x \in X$, then the fuzzy set A is called a *crisp set.*

Definition 2.3.[5] Let X be a set. A *fuzzy topology* $\mathcal{T}$ on X is a subset of $F(X)$ satisfying the following conditions.

1. $\underline{\alpha} \in \mathcal{T}$ for all constant $\alpha \in [0,1]$.
2. if $U_i \in \mathcal{T}$ for all $i \in \Lambda$, then $\bigcup_\Lambda U_i \in \mathcal{T}$.
3. if $U, V \in \mathcal{T}$, then $U \cap V \in \mathcal{T}$.

The pair $(X, \mathcal{T})$ is called a *fuzzy topological space.*

Definition 2.4. A map $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ is called as a *continuous* map if for each $V \in \mathcal{T}'$, $f^{-1}(V) \in \mathcal{T}$, where $f^{-1}(V) = \mu_V \circ f$.

Definition 2.5. Let A be a fuzzy set in a fuzzy topological space $(E, \mathcal{T})$. Then the *interior* of A, denote by $\mathring{A}$, is the union of all open fuzzy sets contained in A:

$$\mathring{A} = \cup\{G : G \subset A \text{ and } G \in \mathcal{T}\}.$$

The *closure* of A, denote by $\bar{A}$, is the intersection of all closed fuzzy sets containing A:

$$\bar{A} = \cap\{F : A \subset F \text{ and } F^C \in \mathcal{T}\}.$$

The *exterior* of A, denote by $\dot{A}$, is the union of all open fuzzy sets contained in A^C:

$$\dot{A} = \cup\{G : G \subset A^C \text{ and } G \in \mathcal{T}\}.$$

Clearly, $\mathring{A}$ and $\dot{A} = (\bar{A})^C$ are in $\mathcal{T}$.

Definition 2.6.[9] Let A be a fuzzy set in a fuzzy topological space $(E, \mathcal{T})$. The *fuzzy boundary* of A, denoted by ∂A, is defined as follows. If $\bar{A} \cap \overline{A^C} = \emptyset$, then $\partial A = \emptyset$. Otherwise, $\partial A = \cap\{\text{closed fuzzy sets } D \text{ in } E : \mu_D(x) = \mu_{\bar{A}}(x) \text{ if } x \in \operatorname{Supp} \bar{A} \cap \overline{A^C}\}$.

Proposition 2.7. [9] *Let A be a fuzzy set in a fuzzy topological space $(E, \mathcal{T})$. Then*

(i) $\bar{A} = \mathring{A} \cup \partial A$.

(ii) $\partial A \supset \bar{A} \cap \overline{A^C}$.

Proposition 2.8. [10] *Let A be a fuzzy set in a fuzzy topological space $(E, \mathcal{T})$. Then $e \in \operatorname{Supp} \partial A$ if $\mu_{\mathring{A}}(e) \neq 1$ and $\mu_{\dot{A}}(e) \neq 1$.*

By the following example, it is known that the converse of Proposition 2.8 is false. Let $E = \{e_1, e_2\}$, let $\mu_A(e_1) = 1, \mu_A(e_2) = \frac{1}{2}$ and $\mathcal{T} = \{\emptyset, E, A\}$. Then $e_1 \in \partial A$ and $\mu_{\mathring{A}}(e_1) = 1$.

Proposition 2.9. [10] *Let A be a fuzzy set in a fuzzy topological space $(E, \mathcal{T})$ and let $e \in \operatorname{Supp} E$. Then $\mu_{\overset{\circ}{A}}(e) \neq 1$ or $\mu_{\dot{A}}(e) \neq 1$.*

3. Fuzzy topological polysystems

Let X be a set and $E = X \times \mathbb{R}$, where $\mathbb{R}$ is the real line. The elements of E may be considered as events. The projection map $\pi : E \to \mathbb{R}$, called a *clock*, denotes the time of an event. Take a fuzzy relation r on E, a fuzzy set in $E \times E$. We assume that r satisfies the following conditions.

1. $r(e, e) = 1$ (reflexive)
2. if $r(e_1, e_2) > 0$ and $r(e_2, e_1) > 0$, then $x = y$ (antisymmetric)
3. $r(e_1, e_2) \geq \bigvee_{e \in E} r(e_1, e) \wedge r(e, e_2)$ (transitive)
4. $r(e_1, e_2) \wedge r(e_1, e_3) \geq r(e_2, e_3) \vee r(e_3, e_2)$ (forward)

Let R be a set of fuzzy relations on E satisfying the above conditions 1-4. The set R is called a *strategy* set and its elements are called *strategies.* We note that $r(e_1, e_2)$ means the degree of the relationship: the event e_2 follows the event e_1.

Definition 3.1. A *fuzzy topological polysystem* is a triple $(E, \mathcal{T}, R)$, where $(E, \mathcal{T})$ is a fuzzy topological space and R is a strategy set such that for all $e_1, e_2 \in E, U \in \mathcal{T}, r \in R$ with $(e_1, e_2) \in \operatorname{Supp} r$ and $e_1 \in \operatorname{Supp} U$, there exists $V \in \mathcal{T}$ such that $e_2 \in \operatorname{Supp} U$ and $\mu_V(e) \leq \bigvee_{e' \in \operatorname{Supp} U} r(e', e)$.

Definition 3.2. A *fuzzy dynamical polysystem* is a fuzzy topological polysystem $(E, \mathcal{T}, R)$ such that $r_1(e_1, e_2) \wedge r_2(e_2, e_3) \leq \bigvee_{r \in R} (r(e_1, e_2) \wedge r(e_2, e_3))$ for any $e_1, e_2 \in E$ and $r_1, r_2 \in R$.

Let $(E, \mathcal{T}, R)$ be a fuzzy dynamical polysystem. Take $e_1, e_2 \in E$ and $r \in R$. The fuzzy set in E, $t(e_1, e_2, r)(e) = r(e_1, e) \wedge r(e, e_2)$, is called the *trajectory* from the event e_1 to the event e_2 with respect to r. For $e_1 \in E$, we define a fuzzy set in E

$$K(e_1)(e) = \bigvee_{r \in R} r(e_1, e),$$

the reachable set from e_1. Let A be a fuzzy set in E. The fuzzy set in E, $K(A) = \bigvee_{e \in \operatorname{Supp} A} K(e)$, is called the *reachable set from A*.

Proposition 3.3. *For all* $e, e_1, e_2 \in E$,

$$K(e)(e_1) \wedge K(e_1)(e_2) \leq K(e)(e_2).$$

Proposition 3.4. *Let* $e_1, e_2 \in E$ *and* $r \in R$. *If* $t(e_1, e_2, r)(e_3) \wedge t(e_1, e_2, r)(e_4) \neq 0$, *then*

$$0 \neq t(e_3, e_4, r) \vee t(e_4, e_3, r) \leq t(e_1, e_2, r).$$

Proposition 3.5. *If* A *is a fuzzy set in* E *such that* $e_1 \in \operatorname{Supp} A, (e_1, e_2) \in \operatorname{Supp} r$ *and* $e_2 \in \operatorname{Supp} \partial K(A)$, *then* $t(e_1, e_2, r) \subseteq \partial K(A)$.

4. Fuzzy control problem

Definition 4.1. Let $(E, \mathcal{T})$ be a fuzzy topological space. A *locally tailed fuzzy set* of $(E, \mathcal{T})$ is a pair (T, H) such that

1. T is a fuzzy set in E.
2. H is a fuzzy relation over T, i.e. $\mu_H(e_1, e_2) \leq \mu_T(e_1) \wedge \mu_T(e_2)$, which is (1) irreflexive, i.e. $\mu_H(e_1, e_2) > 0 \Rightarrow e_1 \neq e_2$, (2) antisymmetric, and (3) transitive.
3. if $e \in \operatorname{Supp} U \cap T$, where $U \in \mathcal{T}$, then there exists $\bar{e} \in \operatorname{Supp} U \cap T$ such that $H(e, \bar{e}) > 0$.

Definition 4.2 Let $(E, \mathcal{T}, R)$ be a fuzzy dynamical polysystem and $\alpha \in [0, 1]$.

1. A *fuzzy control problem* for the system $(E, \mathcal{T}, R)$ is a quintuple $(E, \mathcal{T}, R, S, T)$, where S and T are fuzzy sets in E.
2. An *α-solution* for the fuzzy control problem $(E, \mathcal{T}, R, S, T)$ is a triple (e_1, e_2, r) such that $e_1 \in \operatorname{Supp} S, e_2 \in \operatorname{Supp} T, r \in R$ and $r(e_1, e_2) \geq \alpha$.
3. A *fuzzy optimal control problem* for the control problem $(E, \mathcal{T}, R, S, T)$ is a quintuple $(E, \mathcal{T}, R, S, (T, H))$, where (T, H) is a locally tailed fuzzy set of $(E, \mathcal{T})$.

4. An *optimal α-solution* for the control problem $(E, \mathcal{T}, R, S, (T, H))$ is an α-solution (e_1, e_2, r) of the fuzzy control problem $(E, \mathcal{T}, R, S, T)$ such that there is no α-solution $(\bar{e}, \overline{e_2}, \bar{r})$ of the same fuzzy control problem with $\mu_H(\overline{e_2}, e_2) \geq \alpha$.
5. If (e_1, e_2, r) is an optimal α-solution for a fuzzy optimal control problem, then the trajectory $t(e_1, e_2, r)$ is called an *optimal α-trajectory* for this fuzzy optimal control problem.

Proposition 4.3. *If (e_1, e_2, r) is an optimal α-solution for the fuzzy optimal control problem $(E, \mathcal{T}, R, S, (T, H))$, $e_3 \in \operatorname{Supp} S, r' \in R$ and $r'(e_3, e_2) \geq \alpha$, then (e_3, e_2, r') is an optimal α-solution for $(E, \mathcal{T}, R, S, (T, H))$.*

Proposition 4.4. *If $t(e_1, e_2, r)$ is an optimal α-trajectory for the fuzzy optimal control problem $(E, \mathcal{T}, R, S, (T, H))$ and $e_3 \in (S \cap t(e_1, e_2, r))_\alpha$, then $t(e_1, e_2, r)$ is an optimal α-trajectory for $(E, \mathcal{T}, R, S, (T, H))$.*

Let $(E, \mathcal{T}, R)$ be a fuzzy topological polysystem. Now we assume that if A is a fuzzy set in E with $e_1 \in \operatorname{Supp} A, r(e_1, e_2) > 0$ and $e_2 \in \operatorname{Supp} \partial K(A)$, then $t(e_1, e_2, r) \subset \partial K(A)$.

Proposition 4.5. *If (e_1, e_2, r) is an optimal α-solution of the fuzzy optimal control problem $(E, \mathcal{T}, R, S, (T, H))$, then $t(e_1, e_2, r) \subseteq \partial K(S)$.*

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