QUASI-FUZZY CONTINUITY AND QUASI-FUZZY SEPARATION AXIOMS

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Abstract: The aim of this paper is to study and find characterizations of quasi-fuzzy continuous and quasi-fuzzy closed mappings between fuzzy bitopological spaces. The notion of quasi-fuzzy open sets is used to defined quasi-fuzzy T_i $(i = 0, 1, 2)$ and quasi-fuzzy regular spaces and these spaces are investigated under quasi-fuzzy continuity. Finally, quasi-fuzzy connectednss is introduced and studied to some extent.

1. INTRODUCTION AND PRELIMINARIES

Chang [2] used the concept of fuzzy sets to introduce fuzzy topological spaces and several authors continued the investigation of such spaces. From the fact that there are some non-symmetric fuzzy topological structures, Kubiak [8] first introduced and studied the notion of fuzzy bitopological spaces (A triple (X, τ_1, τ_2) where X is a non-empty set and τ_1 and τ_2 are fuzzy topology on X is called a fuzzy bitopological space (henceforth, fbts for short)), as a natural generalization of fuzzy topological space, and initiated the bitopological aspects due to Kelly [7] in the theory of fuzzy topological spaces. Since then several authors [3,4,6,8-10,12] have contributed to the subsequent development of various fuzzy bitoplogical properties. In Section 2 of this paper we introduce the concept of quasi-fuzzy open (quasi-fuzzy closed) sets as the weaker form of τ_i-fuzzy open (closed) set, and studied its basic properties. In Section 3, we introduce the concepts of quasi-fuzzy continuous, quasi-fuzzy open (closed) mappings on fbts and establish their characteristic properties. In Section 4, we introduce quasi-fuzzy T_i $(i = 0, 1, 2)$ and quasi-fuzzy regular spaces for fbts in terms of quasi Q-neighborhoods and quasi-neighborhoods. Finally, we introduce and study some extent quasi-fuzzy connectednss in fuzzy bitopological setting, in Section 5.

For definitions and results not explained in this paper, we refer to the papers [2,11-13] assuming them to be well known. A fuzzy point in X with support $x \in X$ and value α $(0 < \alpha \leq 1)$ is denoted by x_α. For a fuzzy set A of X, $1 - A$ will stand for the complement of A. By 0_X and 1_X we will mean respectively the constant fuzzy sets taking on the values 0 and 1 on X. A set A of fbts (X, τ_1, τ_2) is called τ_i-fo (τ_i-fc) set if $A \in \tau_i$ $(1 - A \in \tau_i)$. For a fuzzy set A of a fbts (X, τ_1, τ_2),τ_i-int(A) and τ_j-cl(A) means respectively the interior and closure of A with respect to the fuzzy topologies τ_i and τ_j, where indices i and j take values $\{1, 2\}$ and $i \neq j$.

2. QUASI-FUZZY OPEN SETS

Definition 2.1. Let (X, τ_1, τ_2) be a fbts and A be any fuzzy set of X. Then A is called quasi-fuzzy open (briefly, qfo) if for each fuzzy point $x_\alpha \in A$ there exists either a τ_1-fo set U such

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that $x_\alpha \in U \leq A$, or a τ_2-fo set V such that $x_\alpha \in V \leq A$. A fuzzy set A is quasi-fuzzy-closed (briefly, qfc) if the complement $1 - A$ is a qfo set.

Every τ_i-fo (resp. τ_i-fc) set is a qfo (resp. qfc) set but the converses may not be true.

Example 2.2. Let $X = \{a, b, c\}$, $\tau_1 = \{1_X, 0_X, A\}$ and $\tau_2 = \{1_X, 0_X, B\}$ where A and B are fuzzy sets of X given by $A(a) = 0.7, A(b) = 0.4, A(c) = 0.7$; $B(a) = 0.6, B(b) = 0.7, B(c) = 0.3$. We consider a fuzzy set C of X defined by $C(a) = C(b) = C(c) = 0.7$. Then C is a qfo set but neither a τ_1-fo set nor a τ_2-fo set.

Theorem 2.3. *A fuzzy set A of a fbts (X, τ_1, τ_2) is qfo set if and only if it is a union of a τ_1-fo set and a τ_2-fo set.*

Theorem 2.4. (a) *Any union of qfo sets is a qfo set, and*
(b) *any intersection of qfc sets is a qfc set.*

Remark 2.5. The intersection (resp. union) of qfo (resp. qfc) sets need not be a qfo (resp. qfc) set (in Example 2.2, A and B are qfo sets but $A \cap B$ is not a qfo set). However we have:

Theorem 2.6. *Let A, B be fuzzy sets of fbts (X, τ_1, τ_2).*
(a) *If A is a τ_1-fo and τ_2-fo set and B is a qfo set, then $A \cap B$ is a qfo set.*
(b) *If A is a τ_1-cf and τ_2-cf set and B is a qfc set, then $A \cup B$ is a qfc set.*

The following Example shows that the product set of qfo sets need not be a qfo set.

Example 2.7. Let $X = \{a, b, c\}$ and A_k $(k = 1, 2, 3, 4)$ be fuzzy sets of X defined as follows:

$$A_1(a) = 0.7, A_1(b) = 0.4, A_1(c) = 0.7; \quad A_2(a) = 0.3, A_2(b) = 0.4, A_2(c) = 0.7;$$
$$A_3(a) = 0.6, A_3(b) = 0.7, A_3(c) = 0.3; \quad A_4(a) = 0.7, A_4(b) = 0.7, A_4(c) = 0.3.$$

Let $\tau_1 = \{1_X, 0_X, A_1\}$, $\tau_2 = \{1_X, 0_X, A_2\}$, $\sigma_1 = \{1_X, 0_X, A_3\}$ and $\sigma_2 = \{1_X, 0_X, A_4\}$ be fuzzy topologies on X. Then $A = A_1 \cup A_2$ is qfo in (X, τ_1, τ_2) and $B = A_3 \cup A_4$ is qfo in (X, σ_1, σ_2). But $A \times B$ is not qfo in $(X \times X, \tau_1 \times \sigma_1, \tau_2 \times \sigma_2)$.

Theorem 2.7. *Let (X, τ_1, τ_2) and (Y, δ_1, δ_2) be fbts's. If A is a qfo set of X and B is a σ_1-fo and σ_2-fo set in Y, then the product $A \times B$ is a qfo set of the fuzzy product space $(X \times Y, \sigma_1, \sigma_2)$, where σ_k is the fuzzy product topology [9] generated by τ_k and δ_k $(k = 1, 2)$.*

Definition 2.8. A fuzzy set A of a fbts (X, τ_1, τ_2) is called a quasi-Q-nbd (resp. quasi-nbd) of a fuzzy point x_α if there exists a qfo set U such that $x_\alpha q U \leq A$ (resp. $x_\alpha \in U \leq A$).

It is clear that every τ_i-Q-nbd (resp. τ_i-nbd) of a fuzzy point is always a quasi-Q-nbd (resp. quasi-nbd) of the fuzzy point, though not conversely.

Theorem 2.9. *Let A be a fuzzy set of a fbts X. A is a qfo set if and only if it is a quasi-nbd of every fuzzy point $x_\alpha \in A$.*

Definition 2.10. Let A be a fuzzy set of a fbts X.
(a) The quasi-closure of A, denoted by $\mathrm{qcl}(A)$, is defined by

$$\mathrm{qcl}(A) = \cap\{B : A \leq B,\ B \text{ is qfc set }\}.$$

(b) The quasi-interior of A, denoted by $\mathrm{qint}(A)$, is defined by

$$\mathrm{qint}(A) = \cup\{B : B \leq A,\ B \text{ is qfo set }\}.$$

For a fuzzy set A of a fbts X, $\mathrm{qcl}(1 - A) = 1 - \mathrm{qint}(A)$ and A is qfc (resp. qfo) if and only if $A = \mathrm{qcl}(A)$ (resp. $A = \mathrm{qint}(A)$).

Theorem 2.11. *Let A be any fuzzy set of a fbts X. Then $x_\alpha \in qcl(A)$ if and only if for each qfo quasi-Q-nbd U of x_α, UqA.*

Theorem 2.12. *If A is any fuzzy set and B is a qfo set of fbts X with $A\bar{q}B$, then $qcl(A)\bar{q}B$.*

3. QUASI-FUZZY CONTINUOUS AND QUASI-FUZZY OPEN (CLOSED) MAPPINGS

Definition 3.1. Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a mapping. f is called

(a) quasi-fuzzy continuous (briefly, qfc) if $f^{-1}(B)$ is qfo in X for each τ_i-fo set B of Y, equivalently, $f^{-1}(B)$ is qfc in X for each τ_i-fc set B of Y.

(b) quasi-fuzzy open (briefly, qf open) if $f(A)$ is qfo in Y for each τ_i-fo set A of X,

(c) quasi-fuzzy closed (briefly, qf closed) if $f(A)$ is qfc in Y for each τ_i-fc set A of X.

Definition 3.2 [10]**.** A mapping $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is called a fuzzy pairwise continuous (resp. fuzzy pairwise open, fuzzy pairwise closed), briefly, fpc (resp. fp open, fp closed) if the induced mappings $f : (X, \tau_k) \to (Y, \sigma_k)$ are fuzzy continuous [2] (resp. fo, fc [9]).

Remark 3.3. It is clear that every fpc (resp. fp open, fp closed) mapping is qfc (resp. qf open, qf closed). That the converse need not be true is shown by the following Examples.

Example 3.4. Let $X = \{a, b, c\}$ and A_k be fuzzy sets of X defined as follows:

$$A_1(a) = 0.7, A_1(b) = 0.4, A_1(c) = 0.7; \quad A_2(a) = 0.6, A_2(b) = 0.7, A_2(c) = 0.3;$$
$$A_3(a) = 0.3, A_3(b) = 0.7, A_3(c) = 0.6; \quad A_4(a) = 0.7, A_4(b) = 0.7, A_4(c) = 0.7.$$

Let $\tau_1 = \{1_X, 0_X, A_1\}$, $\tau_2 = \{1_X, 0_X, A_2\}$, $\sigma_1 = \{1_X, 0_X, A_3\}$ and $\sigma_2 = \{1_X, 0_X, A_4\}$ be the fuzzy topologies on X.

(a) If $f : (X, \tau_1, \tau_2) \to (X, \sigma_1, \sigma_2)$ is mapping defined by $f(a) = c, f(b) = b, f(c) = a$, then f is qfc mapping but not fuzzy pairwise continuous.

(b) If $g : (X, \sigma_1, \sigma_2) \to (X, \tau_1, \tau_2)$ is a mapping defined by $g(a) = c, g(b) = b, g(c) = a$, then g is qf open mapping but not fuzzy pairwise open.

Now we shall discuss the characteristic properties of qfc, qf open and qf closed mappings in fbts's.

Theorem 3.5. *For a mapping $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ the following are equivalent:*

(a) f is qfc.

(b) For each fuzzy point x_α in X and each σ_i-fo nbd V of $f(x_\alpha)$, there exists a qfo quasi-nbd U of x_α such that $f(U) \leq V$.

(c) For each fuzzy point x_α in X and each σ_i-fo Q-nbd V of $f(x_\alpha)$, there exists a quasi-Q-nbd U of x_α such that $f(U) \leq V$.

(d) For each fuzzy set A of X, $f(qcl(A)) \leq \sigma_i\text{-}cl(f(A))$.

(e) For each fuzzy set B of Y, $qcl(f^{-1}(B)) \leq f^{-1}(\sigma_i\text{-}cl(B))$.

Theorem 3.6. *Let (X, τ_1, τ_2) and (Y, σ_1, σ_2)be fbts's and $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be one to one and onto. Then f is qfc if and only if $f(qint(A)) \leq \sigma_i\text{-}int(f(A))$ for each fuzzy set A of X.*

Theorem 3.7. *Let (X, τ_1, τ_2) and (Y, σ_1, σ_2) be fbts's. If the graph mapping $g : (X, \tau_1, \tau_2) \to (X \times Y, \delta_1, \delta_2)$ of f, where δ_i is the fuzzy product topology generated by τ_i and σ_i (for $i = 1, 2$), defined by $g(x) = (x, f(x))$ for each $x \in X$, is qfc, then f is qfc.*

The product of any two qfc mappings on fbts's need not be qfc.

Example 3.8. Let $X = \{a, b, c\}$ and A_k $(k = 1, 2, 3, 4, 5, 6)$ be fuzzy sets of X defined as follows:

$$A_1(a) = 0.7, A_1(b) = 0.4, A_1(c) = 0.7; \quad A_2(a) = 0.3, A_2(b) = 0.4, A_2(c) = 0.7;$$
$$A_3(a) = 0.3, A_3(b) = 0.5, A_3(c) = 0.2; \quad A_4(a) = 0.3, A_4(b) = 0.5, A_4(c) = 0.7;$$
$$A_5(a) = 0.4, A_5(b) = 0.7, A_5(c) = 0.3; \quad A_6(a) = 0.7, A_6(b) = 0.3, A_6(c) = 0.5.$$

Consider the fuzzy topologies $\tau_1 = \{1_X, 0_X, A_1\}$, $\tau_2 = \{1_X, 0_X, A_2\}$, $\omega_1 = \{1_X, 0_X, A_3\}$ and $\omega_2 = \{1_X, 0_X, A_4\}$, $\sigma_1 = \sigma_2 = \{1_X, 0_X, A_5\}$ and $\eta_1 = \eta_2 = \{1_X, 0_X, A_6\}$ and mappings $f_1 : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ defined by $f_1(a) = c, f_1(b) = a, f_1(c) = b$ and $f_2 : (X, \omega_1, \omega_2) \to (Y, \eta_1, \eta_2)$ defined by $f_2(a) = b, f_2(b) = c, f_2(c) = a$. Then f_1 and f_2 are qfc but the product mapping $f_1 \times f_2 : (X \times X, \tau_1 \times \omega_1, \tau_2 \times \omega_2) \to (X \times X, \sigma_1 \times \eta_1, \sigma_2 \times \eta_2)$ is not qfc

Theorem 3.9. *For a mapping $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following are equivalent:*
(a) *f is qf open.*
(b) *$f(\tau_i\text{-}int(A)) \leq qint(f(A))$ for each fuzzy set A of X.*
(c) *$\tau_i\text{-}int(f^{-1}(B)) \leq f^{-1}(qint(B))$ for each fuzzy set B of Y.*
(d) *For each fuzzy set B of Y and each τ_i-fc set of X such that $f^{-1}(B) \leq A$, there exists a qfc set C of Y such that $B \leq C$ and $f^{-1}(C) \leq A$.*

Theorem 3.10. *For a mapping $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following are equivalent:*
(a) *f is qfc.*
(b) *$f(\tau_i\text{-}cl(A)) \leq qcl(f(A))$ for each fuzzy set A of X.*
(c) *For each fuzzy set B of Y and each τ_i-fo set of X such that $f^{-1}(B) \leq A$, there exists a qfo set C of Y such that $B \leq C$ and $f^{-1}(C) \leq A$.*

Theorem 3.11. *Let (X, τ_1, τ_2) and (Y, σ_1, σ_2)be fbts's and $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be one to one and onto. Then f is qf closed if and only if $f^{-1}(qcl(B)) \leq \tau_i\text{-}cl(f^{-1}(B))$ for each fuzzy set B in Y.*

Theorem 3.12. *Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ and $g : (Y, \sigma_1, \sigma_2) \to (Z, \delta_1, \delta_2)$ be mappings.*
(a) *If f is qfc and $g : (Y, \sigma_i) \to (Z, \delta_i)$ is fuzzy continuous $(i = 1, 2)$, then $g \circ f$ is qfc.*
(b) *If $f : (X, \tau_i) \to (Y, \sigma_i)$ is fuzzy continuous $(i = 1, 2)$ and g is qfc, then $g \circ f$ is qfc.*
(c) *If $f : (X, \tau_i) \to (Y, \sigma_i)$ is fuzzy open (resp. fuzzy closed) $(i = 1, 2)$ and g is qf open (resp. qf closed), then $g \circ f$ is qf open (resp. qf closed).*
(d) *If f is qf open (resp. qf closed) and $g : (Y, \sigma_i) \to (Z, \delta_i)$ is fo (resp. fuzzy closed) $(i = 1, 2)$, then $g \circ f$ is qf open (resp. qf closed).*

4. Quasi separation axioms in fuzzy bitopological spaces

Definition 4.1. A fbts (X, τ_1, τ_2) is quasi-fuzzy T_0 space (briefly, QFT_0) if for any pair of distinct fuzzy point X_α and y_β:

Case I. When $x \neq y$, there exists a qfo set which is quasi-nbd of one of the fuzzy points and not quasi-coincident with the other.

Case II. When $x = y$ and $\alpha < \beta$, there exists a qfo set which is quasi-Q-nbd of y_β and is not quasi-coincident with x_α.

Definition 4.2. A fbts (X, τ_1, τ_2) is quasi-fuzzy T_1 space (briefly, QFT_1) if for any pair of distinct fuzzy points x_α and y_β:

Case I. When $x \neq y$, x_α has a quasi-nbd U and y_β has a quasi-nbd V such that $x_\alpha \bar{q} V$ and $y_\beta \bar{q} U$.

Case II. When $x = y$ and $\alpha < \beta$, y_β has a quasi-Q-nbd V such that $x_\alpha \bar{q} V$.

Definition 4.3. A fbts (X, τ_1, τ_2) is quasi-fuzzy T_2 space (briefly, QFT_2) if for any pair of distinct fuzzy points x_α and y_β:

Case I. When $x \neq y$, x_α has a quasi-nbd U and y_β has a quasi-nbd V such that $U\bar{q}V$.

Case II. When $x = y$ and $\alpha < \beta$, x_α has a quasi-nbd U and y_β has a quasi-Q-nbd V such that $U\bar{q}V$.

Remark 4.4. Obviously, QFT_2 $\Rightarrow$ QFT_1 $\Rightarrow$ QFT_0, but the converses need not be true.

Example 4.5. Let $X = \{a, b, c\}$. We define fuzzy sets of X as follows:

$$B^{ab}_{\alpha\beta}(t) = \begin{cases} \alpha & t = a, \\ \beta & t = b, \\ 0 & t = c, \end{cases} \quad B^{a}_{\alpha}(t) = \begin{cases} \alpha & t = a, \\ 0 & t \in X \setminus \{a\}, \end{cases} \quad B^{a}_{\beta}(t) = \begin{cases} \beta & t = a, \\ 0 & t \in X \setminus \{a\}, \end{cases}$$

where $0 < \alpha \leq \frac{1}{2}$ and $\frac{1}{2} \leq \beta < 1$. Let $\tau_1 = \{1_X, 0_X, B^a_\alpha, B^{ab}_{\alpha\beta}, B^{ac}_{\alpha\beta}, B^{ab}_{\alpha\beta} \cup B^{ac}_{\alpha\beta}\}$ and $\tau_2 = \{1_X, 0_X, B^a_\beta, B^{ba}_{\alpha\beta}, B^{ca}_{\alpha\beta}, B^{ba}_{\alpha\beta} \cup B^{ca}_{\alpha\beta}\}$ be fuzzy topologies on X. Then (X, τ_1, τ_2) is QFT_0 but not QFT_1.

Example 4.6. Let $X = \{a, b, c\}$. We define fuzzy sets of X as follows:

$$B^{a}_{\alpha_1}(t) = \begin{cases} \alpha_1 & t = a, \\ 1 & t = b, \\ 0 & t = c, \end{cases} \quad B^{b}_{\alpha_2}(t) = \begin{cases} 1 & t = a, \\ \alpha_2 & t = b, \\ 0 & t = c, \end{cases} \quad B^{c}_{\alpha_3}(t) = \begin{cases} \alpha_3 & t = c, \\ 0 & t \in X \setminus \{c\}, \end{cases}$$

$$C^{a}_{\beta_1}(t) = \begin{cases} \beta_1 & t = a, \\ 1 & t = b, \\ 0 & t = c, \end{cases} \quad C^{b}_{\beta_2}(t) = \begin{cases} 1 & t = a, \\ \beta_2 & t = b, \\ 0 & t = c, \end{cases} \quad C^{c}_{\beta_3}(t) = \begin{cases} \beta_3 & t = c, \\ 0 & t \in X \setminus \{c\}, \end{cases}$$

where $\frac{1}{2} \leq \alpha_1, \beta_2 \leq 1$, $0 \leq \alpha_2, \beta_1 \leq \frac{1}{2}$ and $0 \leq \alpha_3, \beta_3 \leq 1$. Let $\tau_1 = \{1_X, 0_X, B^a_{\alpha_1}, B^b_{\alpha_2}, B^c_{\alpha_3}, B^a_{\alpha_1} \cap B^b_{\alpha_2}, B^a_{\alpha_1} \cup B^b_{\alpha_2}, B^a_{\alpha_1} \cup B^c_{\alpha_3}, B^b_{\alpha_2} \cup B^c_{\alpha_3}, B^a_{\alpha_1} \cup B^b_{\alpha_2} \cup B^c_{\alpha_3}\}$ and $\tau_2 = \{1_X, 0_X, C^a_{\beta_1}, C^b_{\beta_2}, C^c_{\beta_3}, C^a_{\beta_1} \cap C^b_{\beta_2}, C^a_{\beta_1} \cup C^b_{\beta_2}, C^a_{\beta_1} \cup C^c_{\beta_3}, C^b_{\beta_2} \cup C^c_{\beta_3}, C^a_{\beta_1} \cup C^b_{\beta_2} \cup C^c_{\beta_3}\}$ be fuzzy topologies on X. Then (X, τ_1, τ_2) is QFT_1 but not QFT_2.

Theorem 4.7. *A fbts (X, τ_1, τ_2) is QFT_0 if and only if for any pair of distinct fuzzy points x_α and y_β, either $x_\alpha \notin qcl(y_\beta)$ or $y_\beta \notin qcl(x_\alpha)$.*

Theorem 4.8. *A fbts (X, τ_1, τ_2) is QFT_1 if and only if every fuzzy point x_α is qfc set.*

Theorem 4.9. *A fbts (X, τ_1, τ_2) is QFT_2 if and only if for every fuzzy point x_α in X, $x_\alpha = \cap\{qcl(U) : U$ is quasi-nbd of $x_\alpha\}$ and for every $x, y \in X$ with $x \neq y$, there exists a quasi-nbd U of x_1 such that $y \notin Supp(qcl(U))$.*

Theorem 4.10. *Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be one to one. If f is a qfc mapping and (Y, σ_i) $(i = 1, 2)$ is fuzzy T_k space, then (X, τ_1, τ_2) is QFT_k, for $k = 0, 1, 2$.*

Definition 4.11. A fbts (X, τ_1, τ_2) is quasi-fuzzy regular (QFR) iff for any qfc set F of X and any fuzzy point x_α in X with $x_\alpha \notin F$, there exist qfo sets U and V such that $x_\alpha q U$, $F \leq V$ and $U\bar{q}V$.

Theorem 4.12. *For a fbts (X, τ_1, τ_2), the following statements are equivalent:*

(a) *X is QFR.*

(b) *For any fuzzy set A and any qfc set F with $A \not\leq F$, there exist qfo sets U and V such that AqU, $F \leq V$ and $U\bar{q}V$.*

(c) *For any fuzzy set A and any qfo set U with AqU, there exists a qfo set V such that $AqV \leq qcl(V) \leq U$.*

(d) *Every qfo set V can be expressed as union of qfo sets U_λ's such that $qcl(U_\lambda) \leq V$, for all λ.*

(e) *For each fuzzy point x_α and each qfc set F such that $x_\alpha \notin F$, there exists qfo set U such that $x_\alpha \notin qcl(U)$ and $F \leq U$.*

Now, we characterize QFR space in terms of the quasi-Q-nbd of fuzzy points and the quasi-θ-closure operator.

Definition 4.13. A fuzzy point x_α in a fbts X is said to be a quasi-θ-cluster point of a fuzzy set A if for every quasi-Q-nbd U of x_α, $\text{qcl}(U)\text{q}A$. The set of all quasi-θ-cluster points of A is called the quasi-θ-closure of A and denoted by $\text{qcl}_\theta(A)$. A fuzzy set A of X is quasi-θ-closed if $A = \text{qcl}_\theta(A)$ and the complement of a quasi-θ-closed set is quasi-θ-open.

Lemma 4.14. *Let A, B be fuzzy sets of a fbts (X, τ_1, τ_2). Then we have:*

(a) *$qcl(A) \leq \tau_1\text{-}cl(A)$ and $qcl_\theta(A) \leq \tau_i\text{-}Cl_\theta(A)$.*

(b) *If A is qfo, then $qcl(A) = qcl_\theta(A)$.*

(c) *If $A \leq B$, then $qcl_\theta(A) \leq qcl_\theta(B)$.*

(d) *1_X and 0_X are quasi-θ-closed sets.*

(e) *Every quasi-θ-closed is qfc set.*

Theorem 4.15. *For a fbts (X, τ_1, τ_2), the following statements are equivalent:*

(a) *X is QFR.*

(b) *For each fuzzy point x_α in X and each qfo quasi-Q-nbd U of x_α, there exists a qfo quasi-Q-nbd V of x_α such that $qcl(V) \leq U$.*

(c) *For any fuzzy set A in X, $qcl(A) = qcl_\theta(A)$.*

5. QUASI-FUZZY CONNECTED SETS

Definition 5.1. Two non-null fuzzy sets A and B of a fbts (X, τ_1, τ_2) (i.e., neither A nor B is 0_X) is called quasi-fuzzy separated iff $\text{qcl}(A)\bar{\text{q}}B$ and $\text{qcl}(B)\bar{\text{q}}A$.

Theorem 5.2. *Let A, B be non-null fuzzy sets of a fbts (X, τ_1, τ_2).*

(a) *If A, B are quasi-fuzzy separated, and A_1, B_1 are non-null fuzzy sets such that $A_1 \leq A$ and $B_1 \leq B$, then A_1 and B_1 are also quasi-fuzzy separated.*

(b) *If $A\bar{q}B$ and either both are qfo or both are qfc, then A, B are quasi-fuzzy separated.*

(c) *If A, B are either both qfo or both qfc and if $C_A(B) = A \sqcap (1-B)$ and $C_B(A) = B \sqcap (1-A)$, then $C_A(B)$ and $C_B(A)$ are quasi-fuzzy separated.*

Theorem 5.3. *Two non-null fuzzy sets A and B are quasi-fuzzy separated if and only if there exist two qfo sets U and V such that $A \leq U$, $B \leq V$, $A\bar{q}V$ and $B\bar{q}U$.*

Definition 5.4. A fuzzy set which can not be expressed as the union of two quasi-fuzzy separated sets is said to be a quasi-fuzzy connected set.

Theorem 5.5. *Let A be a non-null quasi-fuzzy connected set of a fbts (X, τ_1, τ_2). If A is contained in the union of two quasi-fuzzy separated sets B and C, then exactly one of the following conditions (a) and (b) holds:*

(a) $A \leq B$ and $A \cap C = 0_X$.

(b) $A \leq C$ and $A \cap B = 0_X$.

Theorem 5.6. *Let $\{A_\alpha | \alpha \in \Lambda\}$ be a collection of quasi-fuzzy connected sets of a fbts X. If there exists $\beta \in \Lambda$ such that $A_\alpha \cap A_\beta \neq 0_X$ for each $\alpha \in \Lambda$, then $A = \cup\{A_\alpha | \alpha \in \Lambda\}$ is quasi-fuzzy connected.*

Theorem 5.7. *Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a qfc surjection. If A is quasi-fuzzy connected, then $f(A)$ is fuzzy connected in (Y, σ_i).*

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