INTUITIONISTIC FUZZY PROXIMITY SPACES

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ABSTRACT. In this paper, we introduce the concept of the intuitionistic fuzzy proximity space as a generalization of a fuzzy proximity space, and investigate some of their properties. Also we study the relations between intuitionistic fuzzy proximity spaces and intuitionistic fuzzy topological spaces.

1. Introduction

As a generalization of fuzzy sets, the concepts of intuitionistic fuzzy sets was introduced by Atanassov [1]. Recently, Çoker and his colleagues [3,4,5] introduced intuitionistic fuzzy topological spaces using intuitionistic fuzzy sets.

In this paper, we introduce the concept of the intuitionistic fuzzy proximity space as a generalization of a fuzzy proximity space, and investigate some of their properties. Also we study the relations between intuitionistic fuzzy proximity spaces and intuitionistic fuzzy topological spaces.

2. Preliminaries

Let X be a nonempty set. An *intuitionistic fuzzy set* A is an ordered pair

$$A = (\mu_A, \gamma_A)$$

where the functions $\mu_A : X \to I$ and $\gamma_A : X \to I$ denote the degree of membership and the degree of nonmembership respectively, and $\mu_A + \gamma_A \le 1$.

Obviously every fuzzy set μ_A on X is an intuitionistic fuzzy set of the form $(\mu_A, 1 - \mu_A)$.

DEFINITION 2.1. ([1]) Let $A = (\mu_A, \gamma_A)$ and $B = (\mu_B, \gamma_B)$ be intuitionistic fuzzy sets on X. Then

(1) $A \subseteq B$ iff $\mu_A \le \mu_B$ and $\gamma_A \ge \gamma_B$.
(2) $A = B$ iff $A \subseteq B$ and $B \subseteq A$.
(3) $A^c = (\gamma_A, \mu_A)$.
(4) $A \cap B = (\mu_A \wedge \mu_B, \gamma_A \vee \gamma_B)$.
(5) $A \cup B = (\mu_A \vee \mu_B, \gamma_A \wedge \gamma_B)$.
(6) $0_{\sim} = (\tilde{0}, \tilde{1})$ and $1_{\sim} = (\tilde{1}, \tilde{0})$.

Let f be a map from a set X to a set Y. Let $A = (\mu_A, \gamma_A)$ be an intuitionistic fuzzy set of X and $B = (\mu_B, \gamma_B)$ an intuitionistic fuzzy set of Y. Then $f^{-1}(B)$ is an intuitionistic fuzzy set of X defined by

$$f^{-1}(B) = (f^{-1}(\mu_B), f^{-1}(\gamma_B))$$

and $f(A)$ is an intuitionistic fuzzy set of Y defined by

$$f(A) = (f(\mu_A), 1 - f(1 - \gamma_A)).$$

For a map $f : X \to Y$, the following results are well-known:

(1) If $A_1 \subseteq A_2$ then $f(A_1) \subseteq f(A_2)$.
(2) If $B_1 \subseteq B_2$ then $f^{-1}(B_1) \subseteq f^{-1}(B_2)$.
(3) $A \subseteq f^{-1}(f(A))$.
(4) If f is one-to-one, then $A = f^{-1}(f(A))$.
(5) $B \supseteq f(f^{-1}(B))$.
(6) If f is onto, then $B = f(f^{-1}(B))$.
(7) $f^{-1}(\bigcup B_i) = \bigcup f^{-1}(B_i)$.
(8) $f^{-1}(\bigcap B_i) = \bigcap f^{-1}(B_i)$.
(9) $f(\bigcup A_i) = \bigcup f(A_i)$.
(10) $f(\bigcap A_i) \subseteq \bigcap f(A_i)$.
(11) If f is one-to-one, then $f(\bigcap A_i) = \bigcap f(A_i)$.
(12) $f^{-1}(0_\sim) = 0_\sim$.
(13) $f^{-1}(1_\sim) = 1_\sim$.
(14) $f(0_\sim) = 0_\sim$.
(15) $f(1_\sim) = 1_\sim$ if f is onto.
(16) $f(A)^c \subseteq f(A^c)$ if f is onto.
(17) $f^{-1}(B)^c = f^{-1}(B^c)$.

DEFINITION 2.2. ([3]) An *intuitionistic fuzzy topology* on X is a family $\mathcal{T}$ of intuitionistic fuzzy sets in X which satisfies the following properties:

(1) $0_\sim, 1_\sim \in \mathcal{T}$.
(2) If $A_1, A_2 \in \mathcal{T}$, then $A_1 \cap A_2 \in \mathcal{T}$.
(3) If $A_i \in \mathcal{T}$ for each i, then $\bigcup A_i \in \mathcal{T}$.

The pair $(X, \mathcal{T})$ is called an *intuitionistic fuzzy topological space.*

Let $(X, \mathcal{T})$ be an intuitionistic fuzzy topological space. Then any member of $\mathcal{T}$ is called *intuitionistic fuzzy open set* of X and the complement *intuitionistic fuzzy closed set.*

DEFINITION 2.3. ([3]) Let $(X, \mathcal{T})$ be an intuitionistic fuzzy topological space and A an intuitionistic fuzzy set in X. Then the *fuzzy closure* is defined by

$$\mathrm{cl}(A) = \bigcap\{F \mid A \subseteq F, F^c \in \mathcal{T}\}$$

and the *fuzzy interior* is defined by

$$\mathrm{int}(A) = \bigcup\{G \mid A \supseteq G, G \in \mathcal{T}\}.$$

DEFINITION 2.4. ([8]) Let $\alpha, \beta \in [0,1]$ and $\alpha + \beta \leq 1$. An *intuitionistic fuzzy point* $x_{(\alpha,\beta)}$ of X is an IFS of X defined by

$$x_{(\alpha,\beta)}(y) = \begin{cases} (\alpha, \beta) & \text{if } \ y = x, \\ (0,1) & \text{if } \ y \neq x. \end{cases}$$

In this case, x is called the *support* of $x_{(\alpha,\beta)}$ and α and β are called the *value* and the *nonvalue* of $x_{(\alpha,\beta)}$, respectively. An intuitionistic fuzzy point $x_{(\alpha,\beta)}$ is said to *belong* to an intuitionistic fuzzy set $A = (\mu_A, \gamma_A)$ of X, denoted by $x_{(\alpha,\beta)} \in A$, if $\alpha \leq \mu_A(x)$ and $\beta \geq \gamma_A(x)$.

Clearly an intuitionistic fuzzy point can be represented by an ordered pair of fuzzy points as follows:

$$x_{(\alpha,\beta)} = (x_\alpha, 1 - x_{1-\beta}).$$

DEFINITION 2.5. ([8]) Let $x_{(\alpha,\beta)}$ be an intuitionistic fuzzy point of an intuitionistic fuzzy topological space $(X, \mathcal{T})$. An intuitionistic fuzzy set A of X is called an *intuitionistic fuzzy neighborhood* of $x_{(\alpha,\beta)}$ if there is an intuitionistic fuzzy open set B in X such that $x_{(\alpha,\beta)} \in B \subseteq A$.

DEFINITION 2.6. ([3]) Let $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ be intuitionistic fuzzy topological spaces. Then a map $f : X \to Y$ is said to be

(1) *continuous* if $f^{-1}(B)$ is an intuitionistic fuzzy open set of X for each intuitionistic fuzzy open set B of Y, or equivalently, $f^{-1}(B)$ is an intuitionistic fuzzy closed set of X for each intuitionistic fuzzy closed set B of Y,
(2) *open* if $f(A)$ is an intuitionistic fuzzy open set of Y for each intuitionistic fuzzy open set A of X,
(3) *closed* if $f(A)$ is an intuitionistic fuzzy closed set of Y for each intuitionistic fuzzy closed set A of X,
(4) a *homeomorphism* if f is bijective, continuous and open.

3. Intuitionistic fuzzy proximity spaces

Let $I(X)$ be a family of all intuitionistic fuzzy sets on X.

DEFINITION 3.1. An *intuitionistic fuzzy proximity* on X is a binary relation δ on $I(X)$ satisfying the following properties:

(1) $A\delta B$ implies $B\delta A$.
(2) $(A\cup B)\delta C$ if and only if $A\delta C$ or $B\delta C$.
(3) $A\delta B$ implies $A\neq 0_\sim$ and $B\neq 0_\sim$.
(4) $A\not\delta B$ implies that there exists an $E\in I(X)$ such that $A\not\delta E$ and $E^c\not\delta B$.
(5) $A\cap B\neq 0_\sim$ implies $A\delta B$.

The pair (X,δ) is called an *intuitionistic fuzzy proximity spaces.*

We have easily the following lemma.

LEMMA 3.2. *Let (X,δ) be an intuitionistic fuzzy proximity space. Then we have the following statements.*

(1) *If $A\delta B$, $A_1\supseteq A$ and $B_1\supseteq B$, then $A_1\delta B_1$.*
(2) *$A\delta A$ for each $A\neq 0_\sim$.*
(3) *$A\delta 1_\sim$ if and only if $A\neq 0_\sim$.*

DEFINITION 3.3. Let (X,δ_1) and (Y,δ_2) be two intuitionistic fuzzy proximity spaces and $f:X\to Y$ a map. Then f is called a *proximity continuous* map if $A\delta_1 B$ implies $f(A)\delta_2 f(B)$.

From $A\subseteq f^{-1}f(A)$ and $C\supseteq ff^{-1}(C)$ we obtain the following lemma.

LEMMA 3.4. *Let (X,δ_1) and (Y,δ_2) be two intuitionistic fuzzy proximity spaces and $f:X\to Y$ a map. Then f is proximity continuous if and only if $C\not\delta_2 D$ implies $f^{-1}(C)\not\delta_1 f^{-1}(D)$ for each $C,D\in I(Y)$.*

THEOREM 3.5. [3] *Let $(X,\mathcal{T})$ be an intuitionistic fuzzy topological space and $\mathrm{cl}:I(X)\to I(X)$ the fuzzy closure in $(X,\mathcal{T})$. Then for $A,B\in I(X)$,*

(1) $\mathrm{cl}(0_\sim)=0_\sim$.
(2) $A\subseteq \mathrm{cl}(A)$.
(3) $\mathrm{cl}(A\cup B)=\mathrm{cl}(A)\cup\mathrm{cl}(B)$.
(4) $\mathrm{cl}(\mathrm{cl}(A))=\mathrm{cl}(A)$.

THEOREM 3.6. *Let $\mathrm{cl}:I(X)\to I(X)$ be a map satisfying (1)-(4) of the above theorem. Then there is an unique intuitionistic fuzzy topology $\mathcal{T}$ on X such that $\mathrm{cl}=\mathrm{cl}_{\mathcal{T}}$.*

THEOREM 3.7. *Let (X, δ) be an intuitionistic fuzzy proximity space and define a map* $\mathrm{cl} : I(X) \to I(X)$ *by*

$$\mathrm{cl}(A) = \bigcap \{B^c \in I(X) \mid A \bar{\delta} B\}$$

for each $A \in I(X)$. Then

(1) $A \subseteq \mathrm{cl}(A)$.
(2) $\mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}(A)$.
(3) $\mathrm{cl}(A \cup B) = \mathrm{cl}(A) \cup \mathrm{cl}(B)$.
(4) $\mathrm{cl}(0_{\sim}) = 0_{\sim}$.

By Theorem 3.6, hence, the family of $I(X)$

$$\mathcal{T}(\delta) = \{A \in I(X) \mid \mathrm{cl}(A^c) = A^c\}$$

is an intuitionistic fuzzy topology on X. We call $\mathcal{T}(\delta)$ the intuitionistic fuzzy topology on X induced by δ.

THEOREM 3.8. *Let (X, δ_1) and (Y, δ_2) be two intuitionistic fuzzy proximity spaces and $f : X \to Y$ a proximity continuous map. Then $f : (X, \mathcal{T}(\delta_1)) \to (Y, \mathcal{T}(\delta_2))$ is continuous with respect to the corresponding intuitionistic fuzzy topologies $\mathcal{T}(\delta_1)$ and $\mathcal{T}(\delta_2)$.*

4. δ-neighborhoods on the intuitionistic fuzzy proximity

In this section, we will introduce the notion of δ-neighborhoods on the intuitionistic fuzzy proximity.

DEFINITION 4.1. Let (X, δ) be an intuitionistic fuzzy proximity space. For $A, B \in I(X)$ we say that B is a *δ-neighborhood* of A (in symbols $A \ll B$) if $A \bar{\delta} B^c$.

Clearly, we have that if $A \ll B$ then $A \subseteq B$.

THEOREM 4.2. *Let (X, δ) be an intuitionistic fuzzy proximity space and $A, B \in I(X)$. Then*

(1) *$A \ll B$ if and only if $\mathrm{cl}(A) \ll B$.*
(2) *$A \ll B$ implies that there exists an element G of the intuitionistic fuzzy topology $\mathcal{T}(\delta)$ induced by δ on X such that $A \subseteq G \subseteq B$.*
(3) *$A \bar{\delta} B$ implies that there are $E, F \in I(X)$ such that $A \ll E$, $B \ll F$ and $E \bar{\delta} F$.*

THEOREM 4.3. *Let (X, δ) be an intuitionistic fuzzy proximity space. Then the binary relation $\ll$ on $I(X)$ has the following properties:*

(1) $1_\sim \ll 1_\sim$.
(2) $A \ll B$ *implies* $A \cap B^c = 0_\sim$.
(3) *If* $A_1 \subseteq A \ll B \subseteq B_1$, *then* $A_1 \ll B_1$.
(4) $A \ll B_1 \cap B_2$ *if and only if* $A \ll B_1$ *and* $A \ll B_2$.
(5) $A \ll B$ *implies* $B^c \ll A^c$.
(6) *If* $A \ll B$, *then there exists a set* $E \in I(X)$ *such that* $A \ll E \ll B$.

THEOREM 4.4. *Let $\ll$ be a binary relation on $I(X)$ satisfying* (1)-(6) *of the above theorem. Then the binary relation δ on $I(X)$, defined by $A \not\delta B$ if and only if $A \ll B^c$, is an intuitionistic fuzzy proximity on X. Also with respect to this intuitionistic fuzzy proximity, B is a δ-neighborhood of A if and only if $A \ll B$.*

THEOREM 4.5. *Let (X, δ) be an intuitionistic fuzzy proximity space and $A \in I(X)$. Then*

$$\mathrm{cl}(A) = \bigcap\{B \mid A \ll B\}.$$

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