비선형 퍼지 미분 시스템의 최적 제어 문제

Optimal control problem for the nonlinear fuzzy differential systems

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1. Introduction and Preliminaries

Let E_N be the set of all upper semicontinuous convex fuzzy numbers with bounded α-level intervals.

The purpose of this note is to investigate the existence of optimal control for the nonlinear fuzzy control system.

$$\text{(F.C.S)} \quad \begin{cases} \dot{x}(t) = a(t)x(t) + f(t, x(t)) + u(t), \\ x(0) = x_0 \end{cases}$$

where $a : [0, T] \to E_N$ is fuzzy coefficient, initial value $x_0 \in E_N$ and $f : [0, T] \times E_N \to E_N$ nonlinear function satisfies a global Lipschitz condition and control function $u(t) \in E_N$.

Given initial state x_0, we seek the control u_0 minimizing

$$J(u) = \frac{1}{2} \int_0^T \| u(t) \|^2 dt$$

while satisfying the terminal in equalities

$$x(T) >_\alpha x^1 \ , \ x^1 \in E_N$$

where $x(T)$ is the value of trajectory of (F.D.S) at time T and x^1 is the target set.

In [10] P. Diamond and P. E. Kloeden proved the fuzzy optimal control for the following system :

$$\dot{x}(t) = a(t)x(t) + u(t), \quad x(0) = x_0$$

where $x(\cdot), u(\cdot) \in E_N$

Recently Young-Chel Kwun and Dong-Gun Park ([13]) proved the fuzzy optimal control for the following system :

$$\dot{x}(t) = a(t)x(t) + f(t) + u(t), \quad x(0) = x_0$$

on assumption that for any given $T > 0$ there exists some compact interval valued function

$M(T) = [M_l(T), M_r(T)]$ such that

$$M(T) = \int_0^T S(T-s)f(s)\,ds, \quad [M(t)]^\alpha = [M_l^\alpha(T), M_r^\alpha(T)].$$

We consider properties of the fuzzy number and metrics.

A fuzzy subset of R^n is defined in terms of a membership function which assigns to each point $x \in R^n$ a grade of membership in the fuzzy set. Such a membership function

$$u : R^n \rightarrow [0,1]$$

is used synonomously to denote the corresponding fuzzy set.

Assumption 1. u maps R^n onto $[0,1]$.

Assumption 2. $[u]^0$ is a bounded subset of R^n.

Assumption 3. u is upper semicontinuous.

Assumption 4. u is fuzzy convex.

We denote by E^n the space of all fuzzy subsets u of R^n which satisfy Assumptions 1-4; that is, normal, fuzzy convex, upper semicontinuous fuzzy sets with bounded supports.

In particular $n=1$, denote by E^1 the space of all fuzzy subsets u of R which satisfy Assumptions 1-4.

A fuzzy number a in real line R is a fuzzy set characterized by a membership function

μ_a as $\mu_a : R \rightarrow [0,1]$.

A fuzzy number a is expressed as $a = \int_{x \in R} \mu_a(x)/x$ with the understanding that

$\mu_a(x) \in [0,1]$ represent the grade of membership of x in a and $\int$ denotes the union of

$\mu_a(x)/x$'s.

A fuzzy number a in R is said to be convex if for any real numbers $x, y, z \in R$ with $x \le y \le z$,

$$\mu_a(y) \ge \min\{\mu_a(x), \mu_a(z)\}.$$

A fuzzy number a in R is called normal if the following holds

$$\max_x \mu_a(x) = 1.$$

Let E_N be the set of all upper semicontinuous convex fuzzy numbers with bounded α-level intervals. This means that if $a \in E_N$ then the α-level set

$$[a]^\alpha = \{x \in R \mid a(x) \geq \alpha,\ 0 < \alpha \leq 1\}$$

is a closed bounded interval which we denote by

$$[a]^\alpha = [a_l^\alpha,\ a_r^\alpha]$$

and there exists a $t_0 \in R$ such that $a(t_0) = 1$.

Two fuzzy numbers a and b are called equal $a = b$, if $a(x) = b(x)$ for all $x \in R$. It follows that

$$a = b \Leftrightarrow [a]^\alpha = [b]^\alpha \quad \text{for all} \quad \alpha \in (0,1]\ .$$

A fuzzy number a may be decomposed into its level sets through the resolution identity

$$a = \int_0^1 \alpha[a]^\alpha\ ,$$

where $\alpha[a]^\alpha$ is the product of a scalar α with the set $[a]^\alpha$ and $\int$ is the union of $[a]^\alpha$'s with α ranging from 0 to 1.

The support Γ_a of a fuzzy number a is defined, as a special case of level set, by the following

$$\Gamma_a = \{x \mid \mu_a(x) > 0\}.$$

A fuzzy number a in R is said to be positive if $0 < a_1 < a_2$ holds for the support $\Gamma_a = [a_1, a_2]$ of a, that is, Γ_a is in the positive real line. Similarly, a is called negative if $a_1 \leq a_2 < 0$ and zero if $a_1 \leq 0 \leq a_2$.

Lemma 1.1. ([12]) If $a, b \in E_N$, then for $\alpha \in (0,1]$,

$$[a+b]^\alpha = [a_l^\alpha + b_l^\alpha,\ a_r^\alpha + b_r^\alpha],$$

$$[a \cdot b]^\alpha = [\min\{a_i^\alpha b_j^\alpha\},\ \max\{a_i^\alpha b_j^\alpha\}] \quad (i, j = l, r),$$

$$[a-b]^\alpha = [a_l^\alpha - b_r^\alpha,\ a_r^\alpha - b_l^\alpha]\ .$$

Lemma 1.2 ([12]) Let $[a_l^\alpha, a_r^\alpha]$, $0 < \alpha \leq 1$, be a given family of nonempty intervals. If (1) $[a_l^\beta, a_r^\beta] \subset [a_l^\alpha, a_r^\alpha]$ for $0 < \alpha \leq \beta$ and (2) $[\lim_{k\to\infty} a_l^{\alpha_k}, \lim_{k\to\infty} a_r^{\alpha_k}] = [a_l^\alpha, a_r^\alpha]$ whenever (α_k) is nondecreasing sequence converging to $\alpha \in (0,1]$, then the family $[a_l^\alpha, a_r^\alpha]$, $0 < \alpha \leq 1$, represents the α-level sets of a fuzzy number $a \in E_N$.

Conversely, if $[a_l^\alpha, a_r^\alpha]$, $0 < \alpha \leq 1$, are the α-level sets of a fuzzy number $a \in E_N$, then the conditions (1) and (2) holds true.

For $1 \leq p < \infty$ we define the L_p-metric δ_p on $K_C{}^n$ by

(1.1) $$\delta_p(A, B) = \left(\int_{S^{n-1}} |s(x, A) - s(x, B)|^p \mu(dx) \right)^{1/p}$$

for all $A, B \in K_C{}^n$, where $s(\cdot, A)$ is the support function of A, μ is Lebesgue measure on the unit sphere S^{n-1} and $K_C{}^n$ consisting of all nonempty compact convex subsets of R^n. The L_p-metric δ_p on $K_C{}^n$, which are, essentially, directly defined in terms of support functions, give rise to another class of metric on E^n.

There are the $L_{p,p}$-metric ρ_p for $1 \le p < \infty$ defined by

$$\rho_p(u, v) = \left(\int_0^1 \delta_p([u]^\alpha, [v]^\alpha)\, d\alpha \right)^{1/p} \tag{1.2}$$

for all $u, v \in E^n$. In view of definition of metric δ_p on $K_C{}^n$, we can rewrite (1.2) more transparently as

$$\rho_p(u, v) = \left(\int_0^1 \int_{S^{n-1}} |s_u(\alpha, q) - s_v(\alpha, q)| \mu(dq) d\alpha \right)^{1/p}. \tag{1.3}$$

2. Fuzzy optimal control

We consider the fuzzy optimal control of the following fuzzy control system:

$$\begin{cases} \dot{x}(t) = a(t)x(t) + f(t, x(t)) + u(t), \\ x(0) = x_0, \end{cases} \tag{F.C.S}$$

with fuzzy coefficient $a: [0, T] \to E_N$, initial value $x_0 \in E_N$ and $f: [0, T] \times E_N \to E_N$ satisfies a global Lipschitz condition and control u in $C([0, T]: E_N)$.

For $u, v \in E_N$

$$\rho_2(u, v)^2 = \int_0^1 \int_{S^{n-1}} |s_u(\beta, x) - s_v(\beta, x)|^2 d\mu(x) d\beta,$$

where $\mu(\cdot)$ is unit Lebesgue measure on S^{n-1}.

In particular, define $\|u\| = \rho_2(u, \{0\})$. Observe that if $n = 1$ and $[u]^\beta = [u_l^\beta, u_r^\beta]$, then

$$\|u\|^2 = \int_0^1 ((u_l^\beta)^2 + (u_r^\beta)^2) d\beta.$$

Our problem is to minimize

$$J(u) = \frac{1}{2} \int_0^T \|u(t)\|^2 dt \tag{2.1}$$

subject to

$$x(T) >_\alpha x^1, \quad x^1 \in E_N. \tag{2.2}$$

The function $t \mapsto a(t)x(t)$ is Lipschitz ,

$$\rho_2(ax, ay) \le \max_{t \in [0, T]} \{ |a_l^\alpha(t)|, |a_r^\alpha(t)| \} \rho_2(x, y).$$

Then (F.C.S) has a unique solution on [0,T] for a given continuous control $u(\cdot)$.

For given u, the trajectory $x(t)$ is represented by

(2.3) $$x(t) = S(t)x_0 + \int_0^t S(t-s)f(s, x(s))\,ds + \int_0^t S(t-s)u(s)\,ds, \quad 0 \le t \le T$$

where $S(t)$ is a fuzzy number

$$[S(t)]^\alpha = [S_l^\alpha(t), S_r^\alpha(t)] = [\exp\{\int_0^t a_l^\alpha(s)ds\}, \exp\{\int_0^t a_r^\alpha(s)ds\}].$$

Write $[x(t)]^\beta = [x_l^\beta(t), x_r^\beta(t)]$, with a like notation for $u(t)$.

Let P be the positive orthant in R^n. For a given $\alpha \in I$, defined $P_\alpha \subset E^n{}_{Lip}$ by

$$P_\alpha = \{u \in E^n{}_{Lip} : [u]^\alpha \subset P\}.$$

If $u \in P_\alpha$, write $u >_\alpha 0$ and if $u -_h v >_\alpha 0$ write $u >_\alpha v$, where $-_h$ is Hukuhara difference and if $u >_\alpha 0$ and only if $u \ge 0$ with necessity α. The positive dual cone of P_α is the closed convex cone $P_\alpha^\oplus \subset E^{n^*}{}_{Lip}$, defined by

$$P_\alpha^\oplus = \{p \in E^{n^*}{}_{Lip} : \langle u, p\rangle \ge 0 \text{ for all } u \in P_\alpha\},$$

where $\langle u, p\rangle = p(u)$ is the value at $u \in E^n{}_{Lip}$ of the linear functional $p: E^n{}_{Lip} \to R$, the space of which is denoted by $E^{n^*}{}_{Lip}$.

The significance of $P_\alpha^\oplus$ in our problem is that Lagrange multipliers for local optimization live in this dual cone. Local necessity coditions for (2.1) and (2.2) are more accessible when there is some notation of differentiability of the uncertain constraint functions $G = x(T) -_h x^1$. Let $\Pi: E^1{}_{Lip} \to C(I \times S^0)$ be the canonical embedding where $S^0 = \{-1, +1\}$. The fuzzy function G is said to be (Frechet) differentiable at ξ_0 if the map $\hat{G} = \Pi \circ G$ is Frechet differentiable at ξ_0. A point ξ_0 is said to be a regular point of the uncertain constraint $G(\xi) >_\alpha 0$ if $G(\xi_0) >_\alpha 0$ and there is $h \in R$ such that $\hat{G}(\xi_0) + D\hat{G}(\xi_0)h >_\alpha 0$. Our constraint function be compact-interval valued, $G(\xi) = [G_l(\xi), G_r(\xi)]$, and $J: R^n \to R$. The support function of $G(\xi)$ is

$$\Pi(G(\xi))(x) = S_{G(\xi)}(x) = \begin{cases} -G_l(\xi) & \text{if } x = -1 \\ +G_r(\xi) & \text{if } x = +1 \end{cases}$$

since $S^0 = \{-1, +1\}$. Then $\Pi \circ G = S_{G(\cdot)}$ is obviously differentiable if and only if G_l, G_r are differentiable, and $S'_{G(\xi)}(-1) = -\nabla G_l(\xi)$, $S'_{G(\xi)}(+1) = \nabla G_r(\xi)$. The element of $P_\alpha^\oplus$ can be seen to be of the form $l_0\lambda_0 + l_{+1}\lambda_{+1} + l_{-1}\lambda_{-1}$, $=\lambda_{-1}(+1)$where l_i are nonnegative constants, the λ_i map S^0 to R and $\lambda_{+1}(-1) = 0$, $\lambda_{+1}(+1) \ge 0$, $\lambda_{-1}(-1) \ge 0$ and $\lambda_0(-1) = \lambda_0(+1) \le 0$. So each element of $P_\alpha^\oplus$ acts like a triple of nonnegative constants $(\lambda_{-1}, \lambda_0, \lambda_{+1})$, $\lambda^*(S_{G(\xi)}(\cdot)) = (\lambda_{-1} - \lambda_0)G_l(\xi) + (\lambda_0 + \lambda_{+1})G_r(\xi)$, which is always nonneagtive since $\lambda_0(G_r(\xi) - G_l(\xi)) \ge 0$. If ξ_0 is a regular point which is a solution to the constrained minimization, the Kuhn-Tucker conditions, namely that there exists $\lambda^* \ge 0$

so that

$$\nabla J(\xi_0) + \lambda^*(S_{G(\xi_0)}(\cdot)) = 0$$

$$\lambda^*(S_{G(\xi_0)}(\cdot)) = 0$$

can be written as

$$\nabla J(\xi_0) + (\lambda_{-1} - \lambda_0)\nabla G_l(\xi_0) + (\lambda_0 + \lambda_{+1})\nabla G_r(\xi_0) = 0$$

$$(\lambda_{-1} - \lambda_0) G_l(\xi_0) + (\lambda_0 + \lambda_{+1}) G_r(\xi_0) = 0.$$

for some nonnegative reals $\lambda_{-1}, \lambda_0, \lambda_{+1}$.

This extends quite naturally to a fuzzy real number constraint with necessity α, as follows : Define the function $G: R^n \to E^1$ by

$$[G(\xi)]^\alpha = [G_l^\alpha(\xi), G_r^\alpha(\xi)],$$

where for each $\xi \in R^n$, $G_l^\alpha(\xi)$ is monotone, nondecreasing in α and $G_r^\alpha(\xi)$ is monotone nonincreasing in α (since $\alpha \le \beta$ implies that $[G(\xi)]^\beta \subset [G(\xi)]^\alpha$). Suppose further that $G_l^\alpha(\cdot)$ and $G_r^\alpha(\cdot)$ are differentiable in ξ for each $\alpha \in I$. Write, for each fixed α, $G(\xi) >_\alpha 0$ if and only if $[G(\xi)]^\alpha \ge 0$. Then, if ξ_0 is a regular point of the constraint $G(\xi) >_\alpha 0$ minimizing $J(u)$, there exist nonnegative real numbers $\lambda_{-1}, \lambda_0, \lambda_{+1}$ satisfying

$$\nabla J(u) + (\lambda_{-1} - \lambda_0)\nabla_\xi G_l^\alpha(\xi_0) + (\lambda_0 + \lambda_{+1})\nabla_\xi G_r^\alpha(\xi_0) = 0$$

$$(\lambda_{-1} - \lambda_0) G_l^\alpha(\xi_0) + (\lambda_0 + \lambda_{+1}) G_r^\alpha(\xi_0) = 0.$$

Theorem 2.1. There exists fuzzy control $u_0(t)$ for the Fuzzy optimal control problem (2.1) and (2.2) such that

$$\begin{aligned} J(u_0) &= \min J(u) \\ &= \frac{1}{2T^2}\int_0^T \int_0^1 [S_l^\beta(T-s)^{-2}\big((x^1)_l^\beta - S_l^\beta(T)x_{0l}^\beta - \int_0^T S_l^\beta(T-t) f_l^\beta(t, x(t))\,dt\big)^2 \\ &\quad + S_r^\beta(T-s)^{-2}\big((x^1)_r^\beta - S_r^\beta(T)x_{0r}^\beta - \int_0^T S_r^\beta(T-t) f_r^\beta(t, x(t))\,dt\big)^2]\,d\beta\,dt \end{aligned}$$

which is attained when

$$L_{-1}(\beta) = \frac{(x^1)_l^\beta - S_l^\beta(T)x_{0l}^\beta - \int_0^T S_l^\beta(T-t) f_l^\beta(t, x(t))dt}{TS_l^\beta(T-s)^2},$$

$$L_{+1}(\beta) = \frac{(x^1)_r^\beta - S_r^\beta(T)x_{0r}^\beta - \int_0^T S_r^\beta(T-t) f_r^\beta(t, x(t))dt}{TS_r^\beta(T-s)^2}.$$

Proof. Omitted.

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