퍼지 미분 방정식의 해의 존재성과 유일성

Existence and uniqueness of the solutions for the nonlinear fuzzy differential equations

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1. Introduction and Preliminaries

Let E_N be the set of all upper semicontinuous convex fuzzy numbers with beunded α-level intervals.

The purpose of this note is to investigate the existence, uniqueness of the fuzzy solutions of following nonlinear fuzzy differential equations.

(F.D.E) $$\begin{cases} \dot{x}(t) = a(t)x(t) + f(t, x(t)) \ , \ \ 0 \le t \le T, \\ x(0) = x_0 \end{cases}$$

where $a : [0, T] \to E_N$ is fuzzy coefficient, initial value $x_0 \in E_N$ and nonlinear function $f : [0, T] \times E_N \to E_N$ satisfies a global Lipschitz condition.

In [12], Seppo Seikkala proved the existence and uniqueness of the fuzzy solution for the following systems:

$$\begin{cases} \dot{x}(t) = f(t, x(t)) \ , \\ x(0) = x_0 \ , \end{cases}$$

where f is a continuous mapping from $R^+ \times R$ into R and x_0 is a fuzzy number.

We consider properties of the fuzzy number and metrics.

A fuzzy subset of R^n is defined in terms of a membership function which assigns to each point $x \in R^n$ a grade of membership in the fuzzy set. Such a membership function

$$u : R^n \rightarrow [0,1]$$

is used synonomously to denote the corresponding fuzzy set.

Assumption 1. u maps R^n onto $[0,1]$.

Assumption 2. $[u]^0$ is a bounded subset of R^n.

Assumption 3. u is upper semicontinuous.

Assumption 4. u is fuzzy convex.

We denote by E^n the space of all fuzzy subsets u of R^n which satisfy Assumptions 1-4; that is, normal, fuzzy convex, upper semicontinuous fuzzy sets with bounded supports.

In particular $n=1$, denote by E^1 the space of all fuzzy subsets u of R which satisfy Assumptions 1-4.

A fuzzy number a in real line R is a fuzzy set characterized by a membership function μ_a as $\mu_a : R \rightarrow [0,1]$.

A fuzzy number a is expressed as $a = \int_{x \in R} \mu_a(x)/x$ with the understanding that $\mu_a(x) \in [0,1]$ represent the grade of membership of x in a and $\int$ denotes the union of $\mu_a(x)/\ x$'s.

A fuzzy number a in R is said to be convex if for any real numbers x , y , $z \in R$ with $x \le y \le z$,

$$\mu_a(y) \ge \min\ \{\mu_a(x)\ ,\ \mu_a(z)\}\ .$$

A fuzzy number a in R is called normal if the following holds

$$\max_x\ \mu_a(x) = 1.$$

Let E_N be the set of all upper semicontinuous convex fuzzy numbers with bounded α-level intervals. This means that if $a \in E_N$ then the α-level set

$$[a]^\alpha = \{\ x \in R \mid a(x) \ge \alpha\ ,\ 0 < \alpha \le 1\}$$

is a closed bounded interval which we denote by

$$[a]^\alpha = [a_l^\alpha\ ,\ a_r^\alpha]$$

and there exists a $t_0 \in R$ such that $a(t_0) = 1$.

Two fuzzy numbers a and b are called equal $a = b$, if $a(x) = b(x)$ for all $x \in R$.

It follows that

$$a = b \Leftrightarrow [a]^\alpha = [b]^\alpha \text{ for all } \alpha \in (0,1] .$$

A fuzzy number a may be decomposed into its level sets through the resolution identity

$$a = \int_0^1 \alpha\, [a]^\alpha ,$$

where $\alpha[a]^\alpha$ is the product of a scalar α with the set $[a]^\alpha$ and $\int$ is the union of 1 $[a]^\alpha$'s with α ranging from 0 to 1.

The support Γ_a of a fuzzy number a is defined, as a special case of level set, by the following

$$\Gamma_a = \{ x \;\; \mu_a(x) > 0 \}.$$

A fuzzy number a in R is said to be positive if $0 < a_1 < a_2$ holds for the support

$\Gamma_a = [a_1, a_2]$ of a, that is, Γ_a is in the positive real line. Similarly, a is called negative if $a_1 \leq a_2 < 0$ and zero if $a_1 \leq 0 \leq a_2$.

Lemma 1.1. ([12]) If a, $b \in E_N$, then for $\alpha \in (0,1]$,

$$[a+b]^\alpha = [a_l^\alpha + b_l^\alpha, a_r^\alpha + b_r^\alpha],$$

$$[a \cdot b]^\alpha = [\min\{ a_i^\alpha b_j^\alpha \},\ \max\{ a_i^\alpha b_j^\alpha \}] \quad (i, j = l, r),$$

$$[a - b]^\alpha = [a_l^\alpha - b_r^\alpha,\ a_r^\alpha - b_l^\alpha].$$

Lemma 1.2 ([12]) Let $[a_l^\alpha, a_r^\alpha]$, $0 < \alpha \leq 1$, be a given family of nonempty intervals. If (1) $[a_l^\beta, a_r^\beta] \subset [a_l^\alpha, a_r^\alpha]$ for $0 < \alpha \leq \beta$ and (2) $[\lim_{k\to\infty} a_l^{\alpha_k} \;\; \lim_{k\to\infty} a_r^{\alpha_k}] = [a_l^\alpha \;\; a_r^\alpha]$ whenever (α_k) is nondecreasing sequence converging to $\alpha \in (0,1]$, then the family $[a_l^\alpha, a_r^\alpha]$, $0 < \alpha \leq 1$, represents the α-level sets of a fuzzy number $a \in E_N$. Conversely, if $[a_l^\alpha, a_r^\alpha]$, $0 < \alpha \leq 1$, are the α-level sets of a fuzzy number $a \in E_N$, then the conditions (1) and (2) holds true.

Let x be a point in R^n and A be a nonempty subset of R^n. We define the distance $d(x, A)$ from x to A by

(1.1) $$d(x, A) = \inf\{ \| x - a \| : a \in A \}.$$

Now let A and B be nonempty subsets of R^n. We define the Hausdorff separation of B from A by

(1.2) $$d_H^*(B, A) = \sup \{ d(b, A) : b \in B \},$$

in general $d_H^*(A,B) \neq d_H^*(B,A)$.

We define the Hausdorff distance between nonempty subsets of A and B of R^n by

(1.3) $$d_H(A,B) = \max \{ d_H^*(A,B), d_H^*(B,A) \}.$$

This is now symmetric in A and B. Consequently

(1) $d_H(A,B) \geq 0$ with $d_H(A,B) = 0$ if and only if $\overline{A} = \overline{B}$.

(2) $d_H(A,B) = d_H(B,A)$

(3) $d_H(A,B) \leq d_H(A,C) + d_H(C,B)$

for any nonempty subsets of A, B and C of R^n. The Hausdorff distance (1.3) is a metric, the Hausdorff metric.

The supremum metric d_∞ on E^n is defined by

(1.4) $$d_\infty(u, v) = \sup\{d_H([u]^\alpha, [v]^\alpha) : \alpha \in (0,1)\} \quad \text{for all } u, v \in E^n$$

and is obviously metric on E^n.

The supremum metric H_1 on $C([0,T] : E^n)$ is defined by

(1.5) $$H_1(x,y) = \sup\{ d_\infty(x(t), y(t)) : t \in [0,T] \} \quad \text{for all } x, y \in C([0,T] : E^n).$$

2. Existence and uniqueness of fuzzy differential equations

We consider the existence and uniqueness of the fuzzy solution for the nonlinear fuzzy differential equation

(F.D.E) $$\begin{cases} \dot{x}(t) = a(t)\, x(t) + f(t, x(t)), \\ x(0) = x_0 \end{cases}$$

with fuzzy coefficient $a : [0,T] \to E_N$, initial value $x_0 \in E_N$ and inhomogeneous term

$f : [0,T] \times E_N \to E_N$ satisfies a global Lipschitz condition, i.e., there exists a finite constant $k > 0$ such that

$$d_H([f(s, \xi_1(s))]^\alpha, [f(s, \xi_2(s))]^\alpha) \leq k\, d_H([\xi_1(s)]^\alpha, [\xi_2(s)]^\alpha)$$

for all $\xi_1(s), \xi_2(s) \in E_N$.

Let I be a real interval. A mapping $x : I \to E_N$ is called a fuzzy process.

We denote

$$[x(t)]^{\alpha} = [x_l^{\alpha}(t), x_r^{\alpha}(t)], \ t \in I, \ 0 < \alpha \leq 1.$$

The derivative $x'(t)$ of a fuzzy process x is defined by

$$[x\ (t)]^{\alpha} = [(x_l^{\alpha})\ (t), (x_r^{\alpha})\ (t)], \ 0 < \alpha \leq 1$$

provided that is equation defines a fuzzy $x'(t) \in E_N$.

The fuzzy integral

$$\int_a^b x(t)dt, \ a, \ b \in I$$

is defined by

$$[\int_a^b x(t)dt]^{\alpha} = [\int_a^b x_l^{\alpha}(t)dt, \int_a^b x_r^{\alpha}(t)dt]$$

provided that the Lebesgue integrals on the right exist.

Definition 2.1. The fuzzy process $x: [0, T] \to E_N$ is a fuzzy solution of the (F.D.E) without inhomogeneous term if and only if

$$(x_l^{\alpha})'(t) = \min\{a_i^{\alpha}(t)x_j^{\alpha}(t) : i, j = l, r\},$$

$$(x_r^{\alpha})'(t) = \max\{a_i^{\alpha}(t)x_j^{\alpha}(t) : i, j = l, r\},$$

$$x_l^{\alpha}(0) = x_{0l}^{\alpha}, \ x_r^{\alpha}(0) = x_{0r}^{\alpha}.$$

Theorem 2.1. For every $x_0 \in E_N$,

$$\begin{cases} \dot{x}(t) = a(t)x(t), \\ x(0) = x_0 \end{cases}$$

has a unique fuzzy solution $x \in C([0, T] : E_N)$.

Proof. Omitted.

Example 2.1. Consider the fuzzy solution of the following fuzzy differential equation

$$\begin{cases} \dot{x} = \tilde{2}\ x, \\ x(0) = \tilde{2} \end{cases}$$

The α-level set of fuzzy number $\tilde{2}$ is $[\tilde{2}]^{\alpha} = [\alpha+1, 3-\alpha]$ for all $\alpha \in [0, 1]$.

From the definition of fuzzy solution,

$$x_l^\alpha(t)=(\alpha+1)\exp\{\int_0^t(\alpha+1)ds\} = (\alpha+1)e^{(\alpha+1)t},$$

$$x_r^\alpha(t)=(3-\alpha)\exp\{\int_0^t(3-\alpha)ds\}=(3-\alpha)e^{(3-\alpha)t}.$$

Therefore

$$[x(t)]^\alpha=[(\alpha+1)e^{(\alpha+1)t},\ (3-\alpha)e^{(3-\alpha)t}] \text{ for all } \alpha\in[0,1].$$

In particular, $[x(t)]^\alpha = \begin{cases} [e^t, 3e^t] & \text{if } \alpha=0 \\ [2e^t, 2e^t] & \text{if } \alpha=1 \end{cases}$

The (F.D.E) is related to the following fuzzy integral equation:

(F.I.E) $$\begin{cases} x(t) = S(t)x_0+\int_0^t S(t-s)f(s,x(s))ds, \\ x(0)=x_0 \in E_N. \end{cases}$$

Theorem 2.2. Let $T>0$, f satisfies a global Lipschitz condition, for every $x_0\in E_N$, (F.D.E) has a unique fuzzy solution $x\in C([0,T]: E_N)$.

Proof. Omitted.

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