제 2 형 퍼지집합들에 대한 퍼지값 기수

Fuzzy-valued cardinality of type 2 fuzzy sets

장 이 채

건국대학교 자연과학대학 응용수학과

충북 충주시 단월동 322

Jang, Lee-Chae

The Department of Applied Mathematics, Kun Kuk University

322 Danwaldong, Chungjusi, Chungbook, KOREA

Abstract

In this paper, we consider generalized concepts of cardinality of a fuzzy sets and obtaind some properties of new concepts of fuzzy-valued cardinality of type 2 fuzzy sets as fuzzy-valued functions. Also, we investigate examples for the calculation of the generalized cardinality of fuzzy-valued functions and compared with concepts of cardinality of a fuzzy set and a fuzzy-valued function.

1. Introduction

A fuzzy set is considered by a function $f : X \to [0,1]$, where X is a set. D. Ralescu [3,4], D. Dubois and H. Prade [1], R. R. Yager [6] investigated concepts of cardinality of a fuzzy set and obtained some properties of new concepts. A statement such as " most students are smart" has a truth value between 0 and 1 ; such a statement is of the general form " Qx's are A" where Q is a fuzzy quantifier and A is a fuzzy subset of a (finite) universe of discourse $X = \{x_1, x_2, \cdots, x_n\}$.

In this paper, we consider a fuzzy-valued function $F : X \to \mathcal{F}(R)$ instead of a fuzzy set A, that is, a real valued function on X. We denote a fuzzy-valued function F by

$$\begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ \mu_{F,a_1} & \mu_{F,a_2} & \cdots & \mu_{F,a_n} \end{pmatrix}$$

Many people have studied the theory of fuzzy sets and fuzzy numbers. We will use the following fuzzy numbers. Let R be the set of real numbers and $[0,1]$ the interval in R. $\mathcal{F}(R)$ denote the set of all fuzzy sets in R.

Definition 1.1 A fuzzy set $A \in \mathcal{F}(R)$ is called a fuzzy number, if and only if

(1) A is normal, i.e. there is at least a $x_0 \in R$ such that $\mu_A(x_0) = 1$.

(2) A is convex, i.e. $\mu_A(\lambda x + (1-\lambda)y) \geq \min(\mu_A(x), \mu_A(y))$ for all $x, y \in R$ and $\lambda \in [0,1]$.

(3) For any $\alpha \in (0,1]$, $A_\alpha = \{x : \mu_A(x) \geq \alpha\}$ is a closed interval and $A_0^- = \{x : \mu_A(x) > 0\}^-$ is compact.

Here, A_0^- is the closure of A_0.

It follows from above that $A \in \mathcal{F}(R)$ is a fuzzy number if and only if there exists a closed interval $[r,s]$ such that

$$\mu_A(x) = \begin{cases} 1 & x \in [r,s] \neq \emptyset \\ L(x) & x < r \\ R(x) & x > s \end{cases}$$

where $L(x)$ is a right continuous function and $R(x)$ is a left continuous function (see [5]).

2. Cardinality and f-cardinality

Let $X = \{x_1, x_2, \cdots, x_n\}$ be a finite set , and let A be a fuzzy subset of X, represented by its membership function $\mu_A : X \longrightarrow [0,1]$. We denote the α-level set of A, $L_\alpha(A) = \{x \in X | \mu_A(x) \geq \alpha\}$ for $0 \leq \alpha \leq 1$. The complement $\bar{A}$ of A has membership function $\mu_{\bar{A}} = 1 - \mu_A$. D. Ralescu[3] and A.L. Ralescu[2] discussed some properties of card$A$, the cardinality of fuzzy set A. This should be a fuzzy subset of $\{0, 1, \cdots, n\}$, with card$A(k)$ being interpreted as the possibility that A has exactly k elements $(0 \leq k \leq n)$. Also, A. L. Ralescu[2] investigated that card$A$ was k to the extent to which exatly k elements of X belong to A while the other $(n-k)$ elements do not belong to A and obtained the following formular, the main formular which can be used to actually calculate card$A(k)$, for each $k = 0, 1, 2, \cdots, n$:

$$\text{card}A(k) = \mu_{(k)} \wedge (1 - \mu_{(k+1)}), \quad k = 0, 1, \cdots, n$$

where $\mu_{(1)}, \mu_{(2)}, \cdots, \mu_{(n)}$ are the values of $\mu_A(x_1), \cdots, \mu_A(x_n)$ arranged in decreasing order of magnitude, and $\mu_{(0)} = 1, \mu_{(n+1)} = 0$.

Now we will consider a new definition of f-card F, the fuzzy-valued cardinality of a fuzzy set-valued function $F : X \rightarrow \mathcal{F}(R)$ with the following membership functions $\mu_{F(x_i)}(x) = \mu_{F,a_i}$ of each $F(x_i)$, for $i = 1, 2, \cdots, n$;

$$\mu_{F,a_i,c_i,d_i}(x) = \begin{cases} 1 & x \in [r_i, s_i] \neq \emptyset \\ L_i(x) & x < r_i \\ R_i(x) & x > s_i \end{cases}$$

where $a_i (i = 1, 2, \cdots, n)$ are some real numbers, $s_i = \frac{m-1}{m} a_i$ and $t_i = \frac{m+1}{m} a_i$, functions L_i are straight lines through two points $(1, s_i), (c_i, 0)$ and functions R_i are straight lines through two points $(1, t_i), (d_i, 0)$, for some numbers c_i, d_i and $i = 1, 2, \cdots, n$ and m is some fixed positive integer. We denote the set of such fuzzy numbers by $\mathcal{F}_0$.

Definition 2.1 The f-cardinality of a fuzzy-valued function F is a fuzzy-valued function f-card F : $\{0, 1, \cdots, n\} \rightarrow \mathcal{F}_0(R)$ by for every k

$$\text{f-card}F(k) = \mu_{F,a_{(k)},c_{(k)},d_{(k)}} \wedge \mu_{F,1-a_{(k+1)},1-d_{(k+1)},1-c_{(k+1)}}$$

where $a_{(0)} \geq a_{(1)} \geq \cdots \geq a_{(n)}$ is the ordered values of the a_i's, $a_{(0)} = 1$ and $a_{(n+1)} = 0$, and $\mu_{F,0}(x) = 1, \mu_{F,1}(x) = 0$ for all $x \in R$. Furthermore, we define the operation $\wedge$ by

$$\mu_{F,a,c,d} \wedge \mu_{F,b,e,f} = \mu_{F,a \wedge b, c \wedge e, d \wedge f}.$$

Notes 2.2 Let us compare with a fuzzy set A (see Example 1 [3]);

$$A = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ 0.9 & 0.8 & 0.1 & 0 \end{pmatrix}$$

Then the cardinality of fuzzy set is

$$\text{card}A = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0.1 & 0.2 & 0.8 & 0.1 & 0 \end{pmatrix}$$

This fact implies that the fuzzy cardinality of fuzzy-valued functions is some generalized definition of cardinality of fuzzy sets, since we consider the fuzzy numbers $\mu_{A,0.9,c_1,d_1}$, $\mu_{A,0.8,c_2,d_2}$, $\mu_{A,0.1,c_3,d_3}$, $\mu_{A,0,0,0}$ instead of the nonfuzzy numbers $0.9, 0.8, 0.1, 0$.

Proposition 2.3 f-card $F(k) = \mathcal{X}_{\{1\}}$ if and only if F is a nonfuzzy set with k elements.

From the definition of the cardinality of fuzzy-valued functions, easily, it implies the following proposition.

Proposition 2.4 For every $k = 0, 1, \cdots, n$, the f-card $F(k)$ of a fuzzy-valued function F lies in $\mathcal{F}_0$.

In this paper, we define the concept of the following complemented fuzzy-valued function $\bar{F}$ of a fuzzy-valued function F

$$\mu_{\bar{F},a_{(i)},c_{(i)},d(i)} = \mu_{F,1-a_{(n-i+1)},1-d_{(n-i+1)},1-c_{(n-i+1)}}$$

for $i = 1, 2, \cdots, n$.

Proposition 2.5 For every fuzzy-valued function F, we have f-card $\bar{F}(i)$= f-card $F(n-i)$

We think that the following result will be used in an aggregation of fuzzy-valued functions, researching in the future. The concepts of an aggregation of fuzzy sets was discussed in D. Ralescu [7]. At first, we define the maximum $\vee$ of elements of $\mathcal{F}_0$.

3. Nonfuzzy f-cardinality

Let $0 \le \alpha \le 1$ and $F : X \to \mathcal{F}_0$ be a fuzzy-valued function. Then the α-level set $[\mu_{F,a,c,d}]_\alpha$ of a fuzzy set $\{(x, \mu_{F,a,c,d}(x)) \mid x \in R\}$ is is denoted by

$$[\mu_{F,a,c,d}]_\alpha = \{x \in R \mid \mu_{F,a,c,d}(x) \ge \alpha\}$$

In this section, we discuss a fuzzy-valued function has a nonfuzzy f-cardinality (number of elements).

Definition 3.1 The nonfuzzy f-cardinality of F, denoted by nf-card F is integer

$$\text{nf-card }_{\alpha_0} F = \begin{cases} 0 & \text{if } F = \emptyset \\ [n_F] & \text{if } F \ne \emptyset \end{cases}$$

where $0 \le \alpha_0 \le 1$, $[n_F]$ is the greatest integer less than n_F and

$$n_F = \begin{cases} j + \mathcal{L}\{x | x \in \left[\mu_{F,a_{(j)},c_{(j)},d_{(j)}}\right]_{\alpha_0}\} & \text{if } a_{(i)} \ge 0.5 \\ (j-1) + \mathcal{L}\{x | x \in \left[\mu_{F,a_{(j-1)},c_{(j-1)},d_{(j-1)}}\right]_{\alpha_0}\} & \text{if } a_{(i)} < 0.5 \end{cases}$$

where $\mathcal{L}(B)$ = the length of an interval B. We note there are uniquely the length of level sets, because they are closed interals.

The following simple procedure can be used to calculate nf-card F.

Algorithm 3.2

(i) If $F = \emptyset$, set nf-card$_{\alpha_0} F = 0$. Stop.

(ii) If $F \neq \emptyset$, find $j = \max\{1 \leq s \leq n | a_{(s-1)} + a_{(s)} > 1\}$.

(iii) Calculate $a_{(j)}$. If $a_{(j)} \geq 0.5$, set $n_F = j + \mathcal{L}\{x | x \in \left[\mu_{F,a_{(j)},c_{(j)},d_{(j)}}\right]_{\alpha_0}\}$.

If $a_{(j)} < 0.5$, set $n_F = (j-1) + \mathcal{L}\{x | x \in \left[\mu_{F,a_{(j-1)},c_{(j-1)},d_{(j-1)}}\right]_{\alpha_0}\}$.

(vi) Calculate $[n_F]$. Set nf-card $F = [n_F]$. Stop.

Discussing the below proposition, we introduce the order of fuzzy-valued sets on X. We define $F \leq G$ by

$$a_{(k)} \leq b_{(k)} \text{ and } \mu_{F,a_{(k)},c_{(k)},d_{(k)}}(x) \leq \mu_{G,b_{(k)},f_{(k)},g_{(k)}}(x - (b_{(k)} - a_{(k)})), \text{ for all } x \in R$$

for $k = 1, 2, \cdots, n$.

Using the definition of order of fuzzy-valued functions, we have the following proposition, an important property of the nonfuzzy f-cardinality.

Proposition 3.3 Let F and G be fuzzy-valued functions on X into $\mathcal{F}_0$. If $F \leq G$, then nf-card $_{\alpha_0}F \leq$ nf-card $_{\alpha_0}G$.

We note that if F is a fuzzy set, the above proposition is agree with proposition 5 [3]. Now, let us consider some examples.

Example 3.4 Let F be a fuzzy-valued function and we denote by

$$F = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ \mu_{F,0.9,-1.1,2.4} & \mu_{F,0.8,-1.0,2.3} & \mu_{F,0.1,-0.9,1.8} & \mu_{F,0,0,0} \end{pmatrix}$$

where $m = 5$. Then we calculated of f-card F as in section 2 with $c_0 = 1, d_0 = 1$ and $c_5 = 0, d_5 = 0$. At first, we find the f_i and g_i, where $f_i = c_i \wedge 1 - d_{i+1}$ and $g_i = d_i \wedge 1 - c_{i+1}$ for $i = 0, 1, \cdots, n$.

$$f_0 = c_0 \wedge 1 - d_1 = 1 \wedge 1 - 2.4 = -1.4 \text{ and } g_0 = d_0 \wedge 1 - c_1 = 1 \wedge 1 - (-1.1) = 2.1$$

So, we obtain $f_0 = -1.4, g_0 = 2.1$ and calculate the remainders by the similar method. Hence we have

$$\text{f-card}_{\alpha_0}F = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ \mu_{F,0.1,\ 1.4,2.1} & \mu_{F,0.2,\ 1.3,2.4} & \mu_{F,0.8,\ 0.8,1.9} & \mu_{F,0.1,\ 0.9,1} & \mu_{F,0,0,0} \end{pmatrix}$$

In this example, we know $j = 2$ and $a_{(2)} > 0.5$. If we put $\alpha_0 = 1$, then $\mathcal{L}\,[\mu_{F,0.8,\ 0.8,1.9}]_1 = 0.32$. So, $n_F = 2.32$ and hence nf-card$_1 F = 2$. But if $\alpha_0 = 0.5$, then we have $\mathcal{L}\,[\mu_{F,0.8,\ 0.8,1.9}]_{0.5} = 1.57$ and $n_F = 3.57$. Therefore we obtain nf-card$_{0.5}F = 3$.

Now we will compare with example 1 in the reference[3]. If $m = 5$ and $\alpha_0 = 1$, we obtained the same number 2 of cradinality of fuzzy set A

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0.9 & 0.8 & 0.1 & 0 \end{pmatrix}$$

But if $m = 5$ and $\alpha_0 = 0.5$, the number 3 is different from the number 2 of of cradinality of fuzzy set A. I guess this definition is the useful tool, because we can deal with some fuzziness of the degree of a membership function of fuzzy set.

References

[1] D. Dubois and H. Prade , Fuzzy cardinality and the modeling of imprecise quantification, Fuzzy Sets and Systems 16 (1985) 199-230.

[2] A.L. Ralescu, A note on rule representation in expert systems, Inform. Sci. 38(1986) 193-203.

[3] Dan Ralescu, Cardinality, quantifier, and the aggregation of fuzzy criteria, Fuzzy Sets and Systems 69 (1995) 355-365.

[4] D. Ralescu, Extensions of fuzzy aggregation, Fuzzy Sets and Systems 86 (1997) 321-330.

[5] Chunhai Yu, Correlation of fuzzy numbers, Fuzzy Sets and Systems 55 (1993) 303-307.

[6] R. R. Yager, Counting the number of classes in a fuzzy set, IEEE Trans. Systems. Man Cybernet. 23 (1993) 257- 264.

[7] L.A. Zadeh, A computational approach to fuzzy quantifiers in natural languages, Comput. Math. 9(1983) 149-184.