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A SOLUTION CONCEPT IN COOPERATIVE FUZZY GAMES

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Abstract
This paper makes a study of the Shapley value in cooperative fuzzy games, games with fuzzy coalitions, which enable the representation of players' participation degrees to each coalition. The Shapley value has so far been introduced only in a class of fuzzy games where a coalition value is not monotone with respect to each player's participation degree. We consider a more natural class of fuzzy games such that a coalition value is monotone with regard to each player's participation degree. The properties of fuzzy games in this class are investigated. Four axioms of Shapley functions are described and a Shapley function of a fuzzy game in the class is given.
Keywords: Cooperative game, fuzzy coalition, the Shapley value, Choquet integral

1. Introduction

The Shapley value [1] [2] is a well known solution in cooperative game theory. Imagine the situation where if some economic agents make up a cooperative relationship, i.e., a coalition, then they can get more gains than that if they do not so. In such cases, one of their interests is how much a agent can get a share from the coalition. The Shapley value is a vector whose element is each player's share which is derived from several reasonable bases.

The Shapley value has been investigated by a number of researchers. Most of them treat games with crisp coalitions. However, there are some situations where some agents do not fully participate in a coalition, but to a certain extent, for example, they offer only a part of resources of their own. Such a coalition can be treated as a so-called fuzzy coalition. The fuzzy coalition, first introduced by Aubin [3] [4], is a collection of economic agents, i.e., players, who transfer fractions of their representability [5] to a specific coalition. Namely, a membership degree shows to what extent a player transfer his/her representability. In this paper, we discuss the Shapley value in fuzzy games, games with fuzzy coalitions.

Butnariu [6] investigated the Shapley value in fuzzy games. He introduced a Shapley function in a limited class of fuzzy games. Given a fuzzy game and a fuzzy coalition, a Shapley value is obtained through a Shapley function. His class is not natural, because the set function $v(S)$ which shows a profit from a coalition S is not monotone nondecreasing with regard to each player's participation degree.

The aim of this paper is to introduce a Shapley function in more natural class of cooperative fuzzy games, which is monotone nondecreasing with regard to each player's participation degree. The class treated in this paper is a set of fuzzy games where a value of each fuzzy coalition can be represented by a Choquet integral of the coalition with respect to the associated crisp game. In such a fuzzy game, a coalition value is monotone with respect to each player's participation degree. The properties of fuzzy games in the class are investigated and a Shapley function is discussed. Finally, an example is given to show a possible application of the proposed Shapley function.

2. Notations and Definitions

In this paper, we consider n-person cooperative fuzzy games with the set of players $N = \{1, \ldots, n\}$. A fuzzy coalition is a fuzzy subset of N identified with a function from N to $[0, 1]$. Then for a fuzzy coalition S and a player i, $S(i)$ indicates the grade of membership of i to S, i.e., the i-th player's participation degree to S. For a fuzzy coalition S, the level set is denoted by $[S]_h = \{i \in N \mid S(i) \geq h\}$. The set of fuzzy coalitions is denoted by $L(N)$. Particularly, $P(N)$ denotes the set of crisp subsets of N. For the sake of simplicity, we define $L(U; N) = \{S \in L(N) \mid S \subseteq U \subseteq N\}$.

An n-person fuzzy game is a function v from $L(N)$ to $\boldsymbol{R}_+ = \{r \in \boldsymbol{R} \mid r \geq 0\}$ such that $v(\emptyset) = 0$. $G(N)$ denotes the set of n-person fuzzy games. Functions from $P(N)$ to $\boldsymbol{R}_+$ are called crisp games, the set of which is denoted by $G_0(N)$. According to the classical interpretation by von Neumann and Morgenstern [7], $v(S)$ is regarded as the least profit when the crisp coalition S is formed. In this interpretation, $v \in G_0(N)$ satisfies the superadditivity. This paper follows this interpretation; thus all $v \in G_0(N)$ treated in this paper are superadditive as an extension of original one.

We introduce the superadditivity into fuzzy games as follows:

Definition 1 *$v \in G(N)$ is said to be superadditive if*

$$v(S \cup T) \geq v(S) + v(T)$$

for any S, $T \in L(N)$ such that $S \cap T = \emptyset$.

We define a payoff function of a fuzzy game based on the idea of imputation in $G_0(N)$:

Definition 2 *A payoff function of a fuzzy game $v \in G(N)$ is a function x from $L(N)$ to $\boldsymbol{R}_+^n$ satisfying the following:*

1. $x_i(U) = 0$ *for any* $i \notin \mathrm{Supp}\ U$,
2. $\sum_{i \in N} x_i(U) = v(U)$,
3. $x_i(U) \geq U(i) \cdot v(\{i\})$,

where $x(U) = (x_1(U), x_2(U), \ldots, x_n(U))$.

Definition 3 *[8], [9] Let $(X, \mathcal{F}, \mu)$ be a fuzzy measure space and g a positive-valued simple function over X, i.e.,*

$$g(r) = \sum_{l=1}^{m} t_l \cdot \chi_{D_l}(r),$$

where $\chi_{D_l}(r) = \begin{cases} 1 & \text{if } r \in D_l \\ 0 & \text{if } r \notin D_l \end{cases}$ $(l = 1, \ldots, m)$. *Then the following is called a Choquet integral of the function g with regard to μ:*

$$(C)\int_X g\, d\mu = \sum_{l=1}^{m} \mu(A_l) \cdot (t_l - t_{l-1})$$

where $0 = t_0 \leq t_1 < \cdots < t_m$ and $A_l = D_l \cup D_{l+1} \cup \cdots \cup D_m$.

3. The Class of Games, $G_C(N)$

In this section, we define a new class of fuzzy games, $G_C(N)$, along with their properties.

Definition 4 *Let $H(S) = \{S(i) \mid S(i) > 0\}$, $q(S)$ be the cardinality of $H(S)$, and rewrite the elements of $H(S)$ in the increasing order as $h_1 < \cdots < h_{q(S)}$. Then v is called a fuzzy game 'with Choquet integral form' iff the following holds for any $S \in L(N)$:*

$$v(S) = \sum_{l=1}^{q(S)} v([S]_{h_l}) \cdot (h_l - h_{l-1}). \qquad (1)$$

We denote by $G_C(N)$ the set of v with Choquet integral forms.

This paper deals with the class of games, $G_C(N)$.

Remark 1 *We should note that $v \in G_C(N)$ in the form of (1) is completely specified by the values $\{v(S) \mid S \in P(N)\}$, i.e., it is derived from a game in $G_0(N)$. In this sense, a fuzzy game with Choquet integral form $v \in G_C(N)$ can be defined by a game $v_0 \in G_0(N)$ and vice versa. For the sake of simplicity, the usual game associated with a fuzzy game with Choquet form $v \in G_C(N)$ is denoted by $v \in G_0(N)$.*

Remark 2 *It is apparent that $(N, 2^N, v)$ is a fuzzy measure space and $S(i)$ is a positive simple function over N. Therefore (1) is a Choquet integral of function S with regard to v.*

Remark 3 *For $v \in G_C(N)$ and $S, T \in L(N)$ such that $S \subseteq T$*

$$v(S) \leq v(T),$$

whenever $v \in G_0(N)$ is superadditive. In other words, if $v \in G_C(N)$ then the function v is nondecreasing with respect to each player's grade of membership.

As described in Section 2, we adopt the classical interpretation of coalition value $v(S)$ by von Neumann and Morgenstern. Since this interpretation lead to the superadditivity of $v \in G_0(N)$, any $v \in G_C(N)$ is nondecreasing with respect to each player's grade of membership in this paper. Moreover, any $v \in G_C(N)$ is supperadditive, as shown in the following proposition.

Proposition 1 *$v \in G_C(N)$ is superadditive iff $v \in G_0(N)$ is superadditive.*

We can find $G_C(N)$ more natural than Butnariu's class of games from the preceding remark.

Lemma 1 *For $v \in G_C(N)$ and $S, T \in L(N)$ such that $S \subseteq T$,*

$$v(S) = v(T) \iff v([S]_h) = v([T]_h), \quad \forall\, h \in (0, 1].$$

4. A $(v : \gamma)$-null Player and a v-carrier in $G_C(N)$

This section is dedicated to showing concepts and properties of a $(v : \gamma)$-null player and those of a v-carrier in $G_C(N)$. $(v : \gamma)$-null players and v-carriers are closely connected with a Shapley function in $G_C(N)$.

4.1. A $(v : \gamma)$-null Player

For some $v \in G(N)$ there may exist a player who cannot contribute to the coalition value further if his participation degree exceeds a certain degree γ. We call him a $(v : \gamma)$-null player, which is an extension of a null player in $G_0(N)$.

Definition 5 *Let $v \in G_C(N)$, $U \in L(N)$ and $S \in L(U; N)$. Let $S_i^U \in L(N)$ be defined by*

$$S_i^U(j) = \begin{cases} U(i), & \text{if } j = i, \\ S(j), & \text{if } j \neq i, \end{cases}$$

If the following is valid, then the player i is called a $(v : \gamma)$-null player in U:

$$v(S) = v(S_i^U), \quad \forall\, S \in L(U; N) \text{ s.t. } S(i) > \gamma.$$

Remark 4 *For $v \in G_C(N)$ and $U \in L(N)$, if the player i is a $(v : \gamma)$-null player in U, then he is a $(v : \gamma')$-null player in U for any $\gamma' \in \{\gamma' \mid \gamma \leq \gamma' \leq U(i)\}$.*

Theorem 1 *Let $v \in G_C(N)$ and $U \in L(N)$. The player i is a $(v:\gamma)$-null player in $U \in L(N)$ iff he/she is a null player in $[U]_h$ for any h ($\gamma < h \leq U(i)$).*

This theorem shows that a $(v:\gamma)$-null player in U corresponds to a usual null player in the level set $[U]_h$ ($\gamma < h \leq U(i)$).

4.2. A v-carrier

We can define a v-carrier as a natural extension of a carrier in $G_0(N)$.

Definition 6 *Let $v \in G_C(N)$, $U \in L(N)$ and $S \in L(U;N)$. If S satisfies the condition, then S is called a v-carrier in U:*

$$v(S \cap T) = v(T), \qquad \forall\, T \in L(U;N).$$

Particularly, if some v-carrier in U is included by any other v-carrier in U, then it is called the smallest v-carrier in U.

Remark 5 *If $v \in G_C(N)$ and $U \in L(N)$, then U is a v-carrier in U.*

Remark 6 *Let $v \in G_C(N)$, $U \in L(N)$ and $S, S' \in L(U;N)$. If $S \supseteq S'$ and S' is a v-carrier in U, then S is also a v-carrier in U.*

Theorem 2 *Let $v \in G_C(N)$, $U \in L(N)$ and $S \in L(U;N)$. S is a v-carrier in U iff $[S]_h$ is a carrier in $[U]_h$ for any $h \in (0,1]$.*

v-carriers in U correspond to carriers in $[U]_h$. Thus we can guarantee the existence of the smallest v-carrier.

Theorem 3 *Let $v \in G_C(N)$. There exists the smallest nonempty v-carrier in U iff $v(\mathrm{Supp}\, U) > 0$.*

The following remark shows a relationship between $(v:\gamma)$-null players and v-carriers.

Remark 7 *Let $v \in G_C(N)$, $U \in L(N)$ and $R \in L(U;N)$. Suppose that R is the smallest v-carrier in U and that there exists a nonempty set of $(v:\gamma)$-null players, which includes the player i. Then we have:*

$$R(i) \leq \gamma < U(i).$$

5. A Shapley Function

The present section deals with a Shapley function based on the original Shapley value [1], [2].

Definition 7 *Let G be a subset of $G(N)$. A Shapley function over G is a function f from G to $(\boldsymbol{R}^n_+)^{L(N)}$ which satisfies the following four axioms.*

Axiom 1 : *If $v \in G$ and $U \in L(N)$, then*

$$\begin{cases} \sum_{i \in N} f_i(v)(U) = v(U) \\ f_i(v)(U) = 0 \quad \forall\, i \notin \mathrm{Supp}\, U. \end{cases}$$

where $f_i(v)(U)$ is the i-th element of $f(v)(U) \in \boldsymbol{R}^n$.

Axiom 2 : *If $v \in G$, $U \in L(N)$, $S \in L(U;N)$ and T is a v-carrier in U, then*

$$f_i(v)(U) = f_i(v)(T).$$

Axiom 3 : *Let $v \in G$, $U \in L(N)$, $R \in L(U;N)$ and $S \in L(R;N)$. Suppose that R is the smallest v-carrier in U. We define $\mathcal{P}_{ij}[S] \in L(N)$ by*

$$\mathcal{P}_{ij}[S](k) = \begin{cases} S(j), & \text{if } k = i, \\ S(i), & \text{if } k = j, \\ S(k), & \text{if } k \neq i, j. \end{cases}$$

If $R(i) = R(j)$ and

$$v(S) = v(\mathcal{P}_{ij}[S]), \qquad \forall\, S \in L(R;N),$$

then

$$f_i(v)(U) = f_j(v)(U).$$

Axiom 4 : *For $v_1, v_2 \in G(N)$, let $v_1 + v_2$ be defined by $(v_1+v_2)(S) = v_1(S) + v_2(S)$ for any $S \in L(N)$. For $U \in L(N)$, if v_1, v_2, $v_1 + v_2 \in G$, then*

$$f_i(v_1+v_2)(U) = f_i(v_1)(U) + f_i(v_2)(U)$$

for any $i \in N$ and any $U \in L(N)$.

5.1. A Shapley Function for $G_0(N)$

If we take $G = G_0(N)$ in Axioms 1 ~ 4, they coincide with the ordinary axioms of the Shapley values in crisp games. Thus the following are immediately found.

Remark 8 *There exists a unique function f' from $G_0(N)$ to $(\boldsymbol{R}^N)^{P(N)}$ such that Axioms 1 ~ 4 are satisfied with f' and $G_0(N)$ instead of f and G, respectively. If $U \in P(N)$ then the function f' can be represented as follows:*

$$f'_i(v)(U) = \begin{cases} \sum_{T \in P_i(U;N)} \beta(|T|;|U|) \cdot (v(T) - v(T \setminus \{i\})) & \text{if } i \in P(U;N) \\ 0 & \text{if } i \notin P(U;N) \end{cases}$$

where $P_i(U;N) = \{T \in P(U;N) \mid T \ni i\}$ and $\beta(t;u) = \dfrac{(t-1)! \cdot (u-t)!}{u!}$ for $T \in P(U;N)$ and $U \in P(N)$.

Remark 9 *For a Shapley function $f'(v)$ over $G_0(N)$ and any game $v \in G_0(N)$, $f'(v)$ is a payoff function of v.*

Remark 10 *If $v \in G_0(N)$ and $S \subseteq T$, then $f'_i(v)(S) \leq f'_i(v)(T)$ for any $i \in N$.*

5.2. A Shapley Function for $G_C(N)$

This subsection deals with a Shapley function for $G_C(N)$. Now we show our main theorem.

Theorem 4 *Let $v \in G_C(N)$ and $U \in L(N)$. The function f defined as follows is a Shapley function over $G_C(N)$:*

$$f_i(v)(U) = \sum_{l=1}^{u}(h_l - h_{l-1}) \cdot f'_i(v)([U]_{h_l}). \qquad (2)$$

Here f' is the function given in Remark 8.

The proof of the above theorem requires the following lemma.

Lemma 2 *The function $f_i(v)(U)$ defined by (2) is non-decreasing with respect to set inclusion, i.e., with respect to each player's participation degree.*

This lemma shows one of important properties of f.

Theorem 5 *For a Shapley function f over $G_C(N)$ and any game $v \in G_C(N)$, $f(v)$ is a payoff function of v.*

The following theorem provides a relationship between the Shapley function values for two strategically equivalent games in $G_C(N)$.

Theorem 6 *For $v, v' \in G_C(N)$, if*

$$v'(S) = c \cdot v(S) + \sum_{i \in N} S(i) \cdot a_i$$

where $c > 0$ and a_i, then

$$f_i(v')(U) = c \cdot f_i(v)(U) + U(i) \cdot a_i \quad \forall\, i \in N.$$

6. Application

Consider three economic companies, named 1, 2 and 3. Company i has 100 units of resource R_i (i=1,2,3). Company i can obtain gains $v(\{i\})$ by producing 100 units of Product P_i from 100 units of Resource R_i. Valuable products can be produced by compounding two and three resources among R_1, R_2 and R_3. Namely, one unit of Product P_{ij} can be produced by compounding one unit of R_i and one unit of R_j ($i < j$, $i, j \in \{1,2,3\}$). Moreover, one unit of Product P_{123} can be produced by compounding one unit of R_1, one unit of R_2 and one unit of R_3. However, to produce Product P_{ij} ($i < j$), Companies i and j have to make up a cooperative relationship, and to produce Product P_{123}, Companies 1, 2 and 3 have to. If Companies i and j make up a full cooperative relationship, i.e., a crisp coalition $\{i, j\}$, then they can obtain gains $v(\{i, j\})$ by producing 100 units of Product P_{ij} ($i < j$). Similarly, by a crisp coalition $\{1, 2, 3\}$, they can obtain gains $v(\{1, 2, 3\})$ by producing 100 units of Product P_{123}. Here, we suppose the superadditivity of v, i.e., for any $S, T \subseteq \{1, 2, 3\}$ s.t. $S \cap T = \emptyset$

$$v(S \cup T) \geq v(S) + v(T).$$

As is in the real life, each company do not need to supply all units of resource the company has to the formed cooperation. Thus, we have to consider a fuzzy game. For example, when Company i can supply only 40 units of R_i to the cooperation between i and j, we regard Player i's participation degree (membership degree) as $0.4 = 40/100$. In such a way, a fuzzy coalition is interpreted. On the other hand, in the setting of this example, the value of a fuzzy coalition can be obtained by Choquet integral. Consider a fuzzy coalition U defined by

$$U(1) = 0.2,\ U(2) = 0.4,\ U(3) = 0.5.$$

This fuzzy coalition means that a cooperation among Companies 1, 2 and 3 is formed and Companies 1, 2 and 3 supply 20, 40 and 50 units of R_1, R_2 and R_3 to the cooperation. Under this cooperation, they can produce 20 units of P_{123}, 20 units of P_{23} and 10 units of P_3. Thus the value of this fuzzy coalition is evaluated by Choquet integral of $U(i)$ with respect to v, i.e.,

$$\begin{aligned} v(U) &= (C)\int_N S\, dv \\ &= \sum_{l=1}^{q(S)} v([S]_{h_l}) \cdot (h_l - h_{l-1}) \\ &= 0.2 \cdot v(\{1,2,3\}) + (0.4 - 0.2) \cdot v(\{2,3\}) \\ &\quad +(0.5 - 0.4) \cdot v(\{3\}) \end{aligned}$$

Now, let us estimate each company's share of $v(U)$ in the fuzzy coalition U. To do this, we can employ the proposed Shapley function. If v is defined by

$$\begin{aligned} &v(\{1\}) = 50, \quad v(\{2\} \cup \{3\}) = 400, \\ &v(\{2\}) = 100, \quad v(\{1\} \cup \{3\}) = 300, \\ &v(\{3\}) = 100, \quad v(\{1\} \cup \{2\}) = 300, \\ &\text{and } v(\{1\} \cup \{2\} \cup \{3\}) = 600, \end{aligned}$$

we obtain $v(U) = 210$.

The ordinary Shapley values are obtained as in Table 1.

Table 1. $f'_i(v)(U)$

U\Company$i$	1	2	3
{1}	50	0	0
{2}	0	100	0
{3}	0	0	100
{1, 2}	87.5	122.5	0
{1, 3}	87.5	0	122.5
{2, 3}	0	140	140
{1, 2, 3}	105	157.5	157.5

The i-th Company's share can be calculated by

$$\begin{aligned} f_i(v)(U) &= 0.2 \cdot f'_i(v)([U]_{0.2}) \\ &\quad +(0.4 - 0.2) \cdot f'_i(v)([U]_{0.4}) \\ &\quad +(0.5 - 0.4) \cdot f'_i(v)([U]_{0.5}) \\ &= 0.2 \cdot f'_i(v)(\{1,2,3\}) \\ &\quad +0.2 \cdot f'_i(v)(\{2,3\}) \\ &\quad +0.1 \cdot f'_i(v)(\{3\}). \end{aligned}$$

Therefore company 1's share can be calculated as follows:

$$\begin{aligned} f_1(v)(U) &= 0.2 \cdot f_1'(v)(\{1,2,3\}) \\ &\quad +0.2 \cdot f_1'(v)(\{2,3\}) \\ &\quad +0.1 \cdot f_1'(v)(\{3\}) \\ &= 30. \end{aligned}$$

In the same way, Companies 2 and 3's shares can be calculated as follows:

$$\begin{aligned} f_2(v)(U) &= 0.2 \cdot f_2'(v)(\{1,2,3\}) \\ &\quad +0.2 \cdot f_2'(v)(\{2,3\}) \\ &\quad +0.1 \cdot f_2'(v)(\{3\}) \\ &= 85, \end{aligned}$$

$$\begin{aligned} f_1(v)(U) &= 0.2 \cdot f_3'(v)(\{1,2,3\}) \\ &\quad +0.2 \cdot f_3'(v)(\{2,3\}) \\ &\quad +0.1 \cdot f_3'(v)(\{3\}) \\ &= 0.2 \cdot f_3'(v)(\{1,2,3\}) \\ &= 95. \end{aligned}$$

7. Conclusion

We have introduced a new class of fuzzy games, $G_C(N)$, and a Shapley function over it, which satisfies the four axioms. The class is more natural than Butnariu's, because any function $v \in G_C(N)$ is nondecreasing with respect to set inclusion, i.e., with respect to each player's participation degree. However, we left the following problems open: (1) continuity of any set function $v \in G_C(N)$ with respect to set inclusion, and (2) uniqueness of the Shapley function for $G_C(N)$.

References

1. L. S. Shapley. "A Value for n-person Games", Annals of Mathematics Studies, **28**, 307-318, 1953.
2. R. J. Aumann, L. S. Shapley. Values of Non-Atomic Games, Princeton University Press, Princeton, 1974.
3. J.-P. Aubin. "Coeur et valeur des jeux flous á paiements lateraux", Comptes Randus de 1' Aoad. Sci. Paris, t.279, s. A. 891-894, 1974.
4. J.-P. Aubin. "Coeur et équilibres des jeux flous sans paiements latéraux", C. R. Acad. Sci. Paris,**279 A**, 963-966, 1974. Comptes Randus de 1' Aoad. Sci. Paris, t.279, s. A. 891-894, 1974.
5. A. Billot. "Fuzzy Convexity and Peripherial Core of an Exchange Economy represented as a Fuzzy Game", in J. Kacprzyk and M. Fedrizzi (Eds.). Multipersons Decision Making Models using Fuzzy Sets and Possibility Thory, Kluwer, Dordrecht, 1990.
6. D. Butnariu. "Stability and Shapley value for an n-persons Fuzzy Game", Fuzzy Sets and Systems, **4**, 63-72, 1980.
7. J. von Neumann, O. Morgenstern. Theory of Games and Economic Behavior, Princeton University Press, 3rd ed., 1953.
8. G. Choquet. "Theory of Capacities", Annales de I'institut Fourier, **5**, 131-295, 1953.
9. M. Sugeno, T. Murofushi. "Choquet's Integral as an Integral Form for a General Class of Fuzzy Measures", Preprints of Second IFSA Congress, **1**, 408-411, 1987.