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Compromise possibility portfolio selections

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Abstract: In this paper, lower and upper possibility distributions are identified to reflect two extreme opinions in portfolio selection problems. Portfolio selection problems based on upper and lower possibility distributions are formalized as quadratic programming problems. Portfolios for compromising two extreme opinions from upper and lower possibility distributions and balancing the opinions of a group of experts can be obtained by quadratic optimization problems, respectively.
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1. Introduction

Multivariate data analysis is a main tool based on probability theory for analyzing the uncertainty in the real world. Possibility data analysis is an alternative based on possibility distributions. Multivariate data analysis considers the uncertainty as probability phenomena while possibility data analysis considers that as possibility phenomena. Possibility theory based on possibility distributions has been proposed by Zadeh [7] and advanced by Dubois and Prade [1]. As an application of possibility theory to portfolio analysis, Tanaka et. al. proposed possibility portfolio selection models [3,4,5]. Although there are some similarities between Markowitz's models and possibility portfolio selection models, these two kinds of models analyze the security data in very different ways. Markowitz's model regards the portfolio selection as probability phenomena so that it minimizes the variance of a portfolio return subject to a given average return [2]. On the contrary, possibility models reflect the experience of portfolio experts, which is characterized by the identified possibility distributions from the given possibility degrees to security data. The basic assumption for using Markowitz's model is that the situation of stock markets in future will be similar to the past one represented by the past security data. It is hard to ensure this kind of assumption for the real ever-changing stock markets. On the other hand, possibility portfolio models integrate security data in past time and experts' knowledge to catch variation of stock markets more feasibly. Because experts' knowledge is very valuable for predicting the future state of stock markets, it is reasonable that possibility portfolio models are useful in the real investment world.

In the paper [4], two kinds of possibility distributions, namely, lower and upper possibility distributions are proposed which are similar to the rough set concept in some sense. Based on these two kinds of distributions, the corresponding portfolio selection models are formalized by quadratic optimization problems minimizing spreads of possibility portfolios subject to the given center returns. It can be concluded that the portfolio return based on a lower possibility distribution has a smaller spread than the one based on an upper possibility distribution. In this sense, two kinds of possibility portfolios based on upper and lower distributions can be considered as conservative and optimistic opinions, respectively. Thus, how to obtain a portfolio for balancing these two extreme opinions is one of topics in this paper. The other topic is to deal with a portfolio selection compromising the opinions of a group of experts.

2. Identification of possibility distributions from the given security data

Let us begin with the given data $(\mathbf{x}_i, h_i)$ $(i=1,...,m)$ where $\mathbf{x}_i=[x_{i1},\cdots,x_{in}]^t$ is a vector of returns of n securities S_j $(j=1,...,n)$ at the ith period and h_i is an associated possibility grade given by expert knowledge to reflect a similarity degree between the future state of stock markets and the state of the ith sample. Assume that these grades h_i $(i=1,...,m)$ are expressed by a possibility distribution **A** defined as

$$\Pi_{\mathbf{A}}(\mathbf{x})=\exp\{-(\mathbf{x}-\mathbf{a})^t\mathbf{D}_{\mathbf{A}}^{-1}(\mathbf{x}-\mathbf{a})\}=(\mathbf{a},\mathbf{D}_{\mathbf{A}})_e, \qquad (1)$$

where $\mathbf{a}=[a_1,a_2,\cdots,a_n]^t$ is a center vector and $\mathbf{D}_{\mathbf{A}}$ is a symmetric positive definite matrix, denoted as $\mathbf{D}_{\mathbf{A}}>0$.

Given the data, the problem is to determine an exponential possibility distribution (1), i.e., a center vector $\mathbf{a}$ and a symmetric positive definite matrix $\mathbf{D}_{\mathbf{A}}$.

The center vector $\mathbf{a}$ can be approximately estimated as

$$\mathbf{a}=\mathbf{x}_{i^*}, \qquad (2)$$

where the grade of $\mathbf{x}_{i^*}$ is $h_{i^*}=\max_{k=1,...,m} h_k$. The associated possibility grade of $\mathbf{x}_{i^*}$ is revised to be 1 because $\mathbf{x}_i$. is regarded as a center vector. Taking the transformation $\mathbf{y} = \mathbf{x} - \mathbf{a}$, the possibility distribution with a zero center vector is represented as,

$$\Pi_{\mathbf{A}}(\mathbf{y}) = \exp(-\mathbf{y}^t \mathbf{D}_{\mathbf{A}}^{-1} \mathbf{y}) = (0, \mathbf{D}_{\mathbf{A}})_e . \quad (3)$$

According to two different viewpoints, two kinds of possibility distributions of **A**, namely, upper and lower possibility distributions are introduced to reflect two kinds of distributions from upper and lower directions. Upper and lower possibility distributions denoted as Π_u and Π_l, respectively with the associated possibility matrices, denoted as $\mathbf{D}_u$ and $\mathbf{D}_l$, respectively should satisfy the inequality $\Pi_u(\mathbf{x}) \geq \Pi_l(\mathbf{x})$. The upper possibility distribution is the one that minimizes the objective function $\Pi_u(\mathbf{y}_1) \times \cdots \times \Pi_u(\mathbf{y}_m)$ subject to the constraint conditions $\Pi_u(\mathbf{y}_i) \geq h_i$ and the lower possibility distribution is the one that maximizes the objective function $\Pi_l(\mathbf{y}_1) \times \cdots \times \Pi_l(\mathbf{y}_m)$ subject to the constraint conditions $\Pi_l(\mathbf{y}_i) \leq h_i$.

In order to ensure that $\Pi_u(\mathbf{y}) \geq \Pi_l(\mathbf{y})$ holds for an arbitrary **y**, the following integrated model with the condition that $\mathbf{D}_u - \mathbf{D}_l$ is a semi-positive definite matrix is introduced.

$$\begin{aligned}
\min_{\mathbf{D}_u, \mathbf{D}_l} \quad & \sum_{i=1}^{m} \mathbf{y}_i^t \mathbf{D}_l^{-1} \mathbf{y}_i - \sum_{i=1}^{m} \mathbf{y}_i^t \mathbf{D}_u^{-1} \mathbf{y}_i \\
\text{subject to} \quad & \mathbf{y}_i^t \mathbf{D}_u^{-1} \mathbf{y}_i \leq -\ln h_i, \\
& \mathbf{y}_i^t \mathbf{D}_l^{-1} \mathbf{y}_i \geq -\ln h_i, \ i=1,\dots,m, \\
& \mathbf{D}_u - \mathbf{D}_l \geq 0, \\
& \mathbf{D}_l > 0. \qquad (4)
\end{aligned}$$

In this case, $\Pi_u(\mathbf{y})$ and $\Pi_l(\mathbf{y})$ are similar to rough set concept shown in Fig. 1. It is obvious that (4) is a nonlinear optimization problem which is difficult to be solved. In order to solve the problem (4) easily, we will use principle component analysis (**PCA**) to rotate the given data ($\mathbf{y}_i, h_i$) to obtain a positive definite matrix. Columns of the transformation matrix **T** are eigenvectors of the matrix $\Sigma = [\sigma_{ij}]$, where σ_{ij} is defined as

$$\sigma_{ij} = \{ \sum_{k=1}^{m} (x_{ki} - a_i)(x_{kj} - a_j) h_k \} / \sum_{k=1}^{m} h_k . \quad (5)$$

Using the linear transformation, the data **y** can be transformed into $\{\mathbf{z} = \mathbf{T}^t \mathbf{y}\}$. Then we have

$$\Pi_{\mathbf{A}}(\mathbf{z}) = \exp\{-\mathbf{z}^t \mathbf{T}^t \mathbf{D}_{\mathbf{A}}^{-1} \mathbf{T} \mathbf{z}\} . \quad (6)$$

According to the feature of **PCA**, $\mathbf{T}^t \mathbf{D}_{\mathbf{A}}^{-1} \mathbf{T}$ is assumed to be a diagonal matrix as follows:

$$\mathbf{T}^t \mathbf{D}_{\mathbf{A}}^{-1} \mathbf{T} = \mathbf{C}_{\mathbf{A}} = \begin{pmatrix} c_1 & & & 0 \\ & \cdot & & \\ & & \cdot & \\ 0 & & & c_n \end{pmatrix} . \quad (7)$$

Denote $\mathbf{C}_{\mathbf{A}}$ as $\mathbf{C}_u$ and $\mathbf{C}_l$ for upper and lower possibility distributions, respectively and denote c_{uj} and c_{lj} (j=1,...,n) as the diagonal elements of $\mathbf{C}_u$ and $\mathbf{C}_l$, respectively. The integrated model can be rewritten as follows.

$$\begin{aligned}
\min_{\mathbf{C}_l, \mathbf{C}_u} \quad & \sum_{i=1}^{m} \mathbf{z}_i^t \mathbf{C}_l \mathbf{z}_i - \sum_{i=1}^{m} \mathbf{z}_i^t \mathbf{C}_u \mathbf{z}_i \\
\text{subject to} \quad & \mathbf{z}_i^t \mathbf{C}_l \mathbf{z}_i \geq -\ln h_i, \\
& \mathbf{z}_i^t \mathbf{C}_u \mathbf{z}_i \leq -\ln h_i, \ i=1,\dots,m, \\
& c_{uj} \geq \varepsilon \\
& c_{lj} \geq c_{uj}, \ j=1,\dots,n, \qquad (8)
\end{aligned}$$

where ε is a very small positive value and the condition $c_{lj} \geq c_{uj} \geq \varepsilon > 0$ makes the matrix $\mathbf{D}_u - \mathbf{D}_l$ semi-positive definite and matrices $\mathbf{D}_u$ and $\mathbf{D}_l$ positive. Thus, we have

$$\mathbf{D}_u = \mathbf{T} \mathbf{C}_u^{-1} \mathbf{T}^t ,$$
$$\mathbf{D}_l = \mathbf{T} \mathbf{C}_l^{-1} \mathbf{T}^t . \quad (9)$$

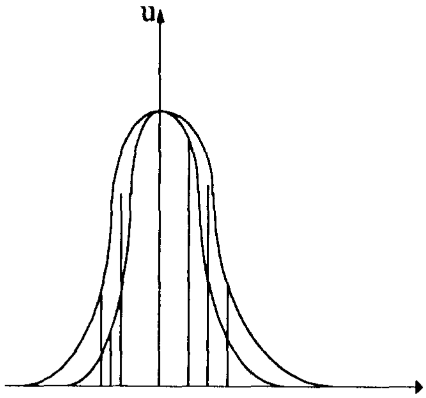

Fig. 1 Upper and lower possibility distributions

3. Possibility portfolio selection models

The portfolio return can be written as

$$z = \mathbf{r}^t \mathbf{x} = \sum_{j=1,\dots,n} r_j x_j , \quad (10)$$

where r_j denotes the proportion of the total investment funds devoted to the security S_j (j=1,...,n) and x_j is its return. Because **x** is governed by a possibility distribution $(\mathbf{a}, \mathbf{D}_{\mathbf{A}})_e$, z becomes a possibility variable Z. Using the extension principle [6], the possibility distribution of a portfolio return Z can be obtained as

$\Pi_Z(z) = \exp\{-(z - \mathbf{r}^t\mathbf{a})^2(\mathbf{r}^t\mathbf{D}_A\mathbf{r})^{-1}\} = (\mathbf{r}^t\mathbf{a}, \mathbf{r}^t\mathbf{D}_A\mathbf{r})_e$
(11)

where $\mathbf{r}^t\mathbf{a}$ is the center value and $\mathbf{r}^t\mathbf{D}_A\mathbf{r}$ is the spread of a portfolio return Z. Given lower and upper possibility distributions, the corresponding portfolio selection models are given as:

Portfolio selection model based on an upper possibility distribution

$$\begin{aligned} \min_{\mathbf{r}} \quad & \mathbf{r}^t\mathbf{D}_u\mathbf{r} \\ \text{subject to} \quad & \mathbf{r}^t\mathbf{a} = c, \\ & \sum_{i=1}^{n} r_i = 1, \\ & r_i \geq 0, \end{aligned} \tag{12}$$

Portfolio selection model based on a lower possibility distribution

$$\begin{aligned} \min_{\mathbf{r}} \quad & \mathbf{r}^t\mathbf{D}_l\mathbf{r} \\ \text{subject to} \quad & \mathbf{r}^t\mathbf{a} = c, \\ & \sum_{i=1}^{n} r_i = 1, \\ & r_i \geq 0, \end{aligned} \tag{13}$$

where c is an expected center value of possibility portfolio return. It is straightforward that models (12) and (13) are quadratic programming problems minimizing the spreads of possibility portfolio returns.

4. Compromised portfolio by upper and lower possibility distributions

The portfolios based on upper and lower possibility distributions are regarded as two extreme viewpoints to reflect the pessimistic and optimistic perspectives because the spread of the portfolio return from the model (12) is larger than the one from (13) for the same center value[4]. Now, we concentrate on how to harmonize these two extreme opinions.

A multi-objective optimization problem which integrates models (12) and (13) is given as follows:

$$\begin{aligned} \min_{\mathbf{r}} \quad & \lambda_1\mathbf{r}^t\mathbf{D}_u\mathbf{r} + \lambda_2\mathbf{r}^t\mathbf{D}_l\mathbf{r}^t \\ \text{subject to} \quad & \mathbf{r}^t\mathbf{a} = c, \\ & \sum_{i=1}^{n} r_i = 1, \\ & r_i \geq 0, \\ & \lambda_1 + \lambda_2 = 1, \\ & \lambda_1 \geq 0, \\ & \lambda_2 \geq 0, \end{aligned} \tag{14}$$

where the weights λ_1 and λ_2 are decided by decision-makers to reflect some preference. Here, the α-method for multi-objective optimization problems is used to decide λ_1 and λ_2 as follows:

Step 1: Solve two quadratic programming problems (12) and (13), where the optimization solutions of (12) and (13) are denoted $\mathbf{r}_u$ and $\mathbf{r}_l$, respectively.

Step 2: Decide the weights λ_1 and λ_2 as follow:

$$\lambda_1 = \varepsilon_1/(\varepsilon_1 + \varepsilon_2) \text{ and } \lambda_2 = \varepsilon_2/(\varepsilon_1 + \varepsilon_2) \tag{15}$$

where $\varepsilon_1 = \mathbf{r}_u{}^t\mathbf{D}_l\mathbf{r}_u - \mathbf{r}_l{}^t\mathbf{D}_l\mathbf{r}_l$ and $\varepsilon_2 = \mathbf{r}_l{}^t\mathbf{D}_u\mathbf{r}_l - \mathbf{r}_u{}^t\mathbf{D}_u\mathbf{r}_u$.

Step 3:Sovle the problem (14) with the obtained weights.

5. Compromised portfolios by a group of experts

Assume that there are k possibility distributions from k experts, denoted as $(\mathbf{a}_i, \mathbf{D}_i)_e$ (i=1,…,k) where $\mathbf{D}_i$ is either the upper or the lower distributions from the ith expert. Now we focus on how to find out a compromised solution from these experts.

Firstly, we consider a portfolio selection from the individual viewpoint of each expert. It leads to a set of QP problems for i=1,…,k as follows:

$$\begin{aligned} \min_{\mathbf{r}} \quad & \mathbf{r}^t\mathbf{D}_i\mathbf{r} \\ \text{subject to} \quad & \mathbf{r}^t\mathbf{a}_i = c, \\ & \mathbf{r}^t\mathbf{I} = 1, \\ & r_j \geq 0. \ (j=1,\ldots,n) \end{aligned} \tag{16}$$

Then, we integrate the above k QP problems into an optimization problem as follows:

$$\begin{aligned} \min_{\mathbf{r}} \quad & \sum_{i=1,\ldots,k} \lambda_i\mathbf{r}^t\mathbf{D}_i\mathbf{r} \\ \text{subject to} \quad & \mathbf{r}^t\mathbf{a}_i \geq c, \ (i=1,\ldots,k), \\ & \mathbf{r}^t\mathbf{I} = 1, \\ & r_j \geq 0, \ (j=1,\ldots,n), \\ & \sum_{i=1,\ldots,k} \lambda_i = 1, \end{aligned} \tag{17}$$

where the weights $\lambda_i \geq 0$ (i=1,…,k) are decided by decision-makers to reflect which expert's opinion is more important than the opinions of others.

The solution from the model (17) can be regarded as a compromise on the investment risks from all of experts' viewpoints, in other words, a strategy to minimize the total investment risks with regard to k experts. The model (17) ensures that the center value of the obtained portfolio is lager than c in any case by its first constraint condition. Thus, it is too restricted in some sense.

Now, we consider another compromise strategy. From the formulation (11), k experts can give k possibility portfolio returns $Z_i = (\mathbf{r}^t\mathbf{a}_i, \mathbf{r}^t\mathbf{D}_i\mathbf{r})_e$, ($i$=1,…,$k$). Let us give Z_i a weight λ_i for representing a confidence level for the ith expert and $\sum_{i=1,\ldots,k} \lambda_i = 1$. A weighed possibility portfolio return is represented as

$$Z = \lambda_1 Z_1 + \ldots + \lambda_k Z_k. \tag{18}$$

Using the extension principle [6], the possibility distribution of Z is obtained as

$$\Pi_Z(z) = (\sum_{i=1,\dots,k} \lambda_i \mathbf{r}^t \mathbf{a}_i, \sum_{i=1,\dots,k} \lambda_i^2 \mathbf{r}^t \mathbf{D}_i \mathbf{r})_e . \quad (19)$$

Thus, the problem for finding out a compromised portfolio is formalized as follows:

$$\begin{aligned} \min_{\mathbf{r}} \quad & \sum_{i=1,\dots,k} \lambda_i^2 \mathbf{r}^t \mathbf{D}_i \mathbf{r}) \\ \text{subject to} \quad & \sum_{i=1,\dots,k} \lambda_i \mathbf{r}^t \mathbf{a}_i = c , \\ & \mathbf{r}^t \mathbf{I} = 1 , \\ & r_j \geq 0 , (j=1,\dots,n), \\ & \sum_{i=1,\dots,k} \lambda_i = 1 . \end{aligned} \quad (20)$$

Compared with the model (17), the solution from the model (20) can be regarded as a compromise on possibility distributions of portfolio returns with respect to k experts. If the parameter λ_i takes 0, the opinion of the ith expert will be completely ignored. With the parameter λ_i increasing, the portfolio obtained form the model (20) will become more similar to the portfolio selected by the ith expert. If the parameter λ_i takes 1, the model (20) only considers the opinion of the ith expert. On the contrary, in the model (17) the center value of the possibility portfolio return from each expert is always considered, simultaneously, even if 0 or 1 is taken for λ_i.

6. Conclusions

This paper proposes an integrated model for obtaining two kinds of possibility distributions, i.e., upper and lower possibility distributions to reflect the different viewpoints of experts in portfolio selection problems. Based on these two kinds of distributions, a portfolio to balance upper and lower possibility distributions is obtained by bi-objective quadratic problems. A portfolio to balance opinions from a group of experts can be obtained by a quadratic optimization problem from two kinds of viewpoints. It can be said compromised possibility portfolios can shorten gaps between upper and lower possibility portfolios or gaps between a group of experts. The existence of multi-distributions in possibility portfolio selection problems is also a distinct point from probability one because there is only one probability distribution in probability portfolio theory. Portfolio selecting based on multi-distributions is one of our research topics in future.

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