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MULTISET-VALUED IMAGES OF FUZZY SETS

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Abstract: An image of a set that produces a multiset from an ordinary set and its extension to fuzzy multisets is considered. For each input element, its image is added to the output regardless whether or not there already exists the same image in the output. Theoretical properties such as commutativity of the image with α-cut or multiset addition are proved. Generalization to the image by multivariable functions is moreover defined.

Keywords: fuzzy multiset, multiset-valued image, α-cut.

1 Introduction

Multisets that are also called bags [3, 6] have weaker mathematical properties than ordinary sets, and therefore have not extensively been studied theoretically. Nevertheless, multisets are commonly observed in various processes in information systems. For example, a standard language for relational database actually handles multisets [2].

Fuzzy multisets have been proposed by Yager [13] and several authors have studied its theory and applications [4, 5, 10, 11, 12, 14], and recently, the authors [8, 9] have found new basic relations and operations by using the notion of the membership sequence for each element, whereby new aspects have been opened in fuzzy multiset theory.

In this paper a new image that produces a multiset from an ordinary set is proposed. This image naturally arises in information processing, as we will see below. Theoretical properties of the image herein is shown to contrast them with those for the ordinary image.

2 Multiset-valued image

Assume first that all universal sets considered herein are finite. Let $X = \{x_1, ..., x_m\}$ and $Y = \{y_1, ..., y_n\}$ be two universal sets and assume a mapping $f: X \to Y$. Let us consider a simple example by which the idea of the new image is introduced.

Example 1. Let $X = \{x_1, ..., x_5\}$, $Y = \{y_1, ..., y_5\}$ and f be given by $f(x_1) = f(x_2) = f(x_3) = y_1$, $f(x_4) = f(x_5) = y_2$ Assume $A = \{x_1, x_2, x_3, x_4\}$, then $f(A) = \{y_1, y_2\}$. Let us consider the following simple procedure:

> " Take elements in A one by one; apply f and output the resulting symbol for the element sequentially, regardless whether or not the symbol already exists in the collection of output symbols."

When applied to the above example, the output is y_1, y_1, y_1, y_2 for the input $A = \{x_1, x_2, x_3, x_4\}$. Since the input is a set, the output should be identical for the change of the order of the input, e.g., x_1, x_2, x_4, x_3. Thus, the output should be a multiset.

The above procedure is interpreted as a multiset-valued mapping defined on the collection of all subsets of X. It is denoted by $f[A]$, in order to distinguish it from the ordinary image $f(A)$. $f[A]$ is called a multiset-valued image.

For the above example with $A = \{x_1, x_2, x_3, x_4\}$,

$$\begin{aligned} f[A] &= \{y_1, y_1, y_1, y_2\} \\ &= \{3/y_1, 1/y_2\} \end{aligned} \tag{1}$$

whereas $f(A) = \{y_1, y_2\}$.

Remark: The symbol $\{\cdot\}$ is used for both ordinary sets and multisets. Note also that the right hand expression of (1) uses $C_B(y)$ which means the number of symbols of y in B. Generally,

$$B = \{C_B(y_i)/y_i\}_{i=1,...,m}.$$

A simple method of calculating the ordinary image $f(A)$ in the following uses $f[A]$:

1. Apply f sequentially to each $x \in A$ and output $f(x)$. (Thus, $f[A]$ is obtained.)

2. For more than one occurrence of a symbol, say $y, ..., y$ in $f[A]$, reduce them into one occurrence: y.

The mapping defined by the above step 2 is denoted by $\mathcal{P}$. $\mathcal{P}$ is defined on the class of all multisets of Y onto the class of all ordinary sets of Y. $\mathcal{P}(f[A]) = \{y_1, y_2\}$ in the above example. The mapping $\mathcal{P}$ is called a projection here. Thus, we have

$$f(A) = \mathcal{P}(f[A]) \tag{2}$$

Notice also that

$$f[A] = \bigoplus_{x \in A} f(x) \tag{3}$$

using the multiset addition $\oplus$, while the ordinary image is

$$f(A) = \bigcup_{x \in A} f(x) \tag{4}$$

using the union $\cup$.

It is straightforward to generalize the multiset-valued mapping $f[\cdot]$ to the domain of fuzzy sets and fuzzy multisets. Let

$$A = \{(x, \mu), ..., (x', \mu')\}$$

be a fuzzy multiset of X, i.e., an arbitrary pair (x, μ) and (x', μ') may or may not be identical. Thus,

$$f[A] = \{(f(x), \mu), ..., (f(x'), \mu')\} \tag{5}$$

The authors have introduced the membership sequence [8]. Namely, for a given fuzzy multiset $B = \{(y, \nu), ..., (y', \nu')\}$ of Y, the membership sequence for an element $y_i \in Y$ is the sequence made from of the collection of all memberships for y_i, arranged into the decreasing order: The membership sequence is denoted by

$$\mu_B^1(y_i), ..., \mu_B^p(y_i)$$

$(\mu_B^1(y_i) \geq \cdots \geq \mu_B^p(y_i))$. The basic operations of the union and intersection are defined using the membership sequence. Appendix A provides a brief review of these operations.

Using the membership sequence, the projection $\mathcal{P}$ is defined:

$$\mu_{\mathcal{P}(B)}(y_i) = \mu_B^1(y_i). \tag{6}$$

In contrast with $f[A]$, we define $f(A)$ for a fuzzy multiset to be $f(\mathcal{P}(A))$: It is obvious to see that $f(\mathcal{P}(A))$ is defined by using the extension principle. Thus, $f(A)$ is an ordinary fuzzy set regardless whether A is a multiset or not.

We have the following propositions, of which the proofs are given in Appendix B.

Proposition 1.

$$f(\mathcal{P}(A)) = \mathcal{P}(f[A]).$$

Example 2.

$$A = \{(x_1, 0.1), (x_1, 0.3), (x_4, 0.5), (x_4, 0.5), (x_5, 0.4)\}$$

is a fuzzy multiset. Using f in Example 1,

$$f[A] = \{(y_1, 0.1), (y_1, 0.3), (y_2, 0.5), (y_2, 0.5), (y_2, 0.4)\}.$$

Moreover,

$$\mathcal{P}(A) = \{0.3/x_1, 0.5/x_4, 0.4/x_5\}.$$

Proposition 2.(Commutativity with the α-cut) Let A be a fuzzy multiset of X. For an arbitrary $\alpha \in (0, 1]$,

$$f[A_\alpha] = (f[A])_\alpha$$

and

$$\mathcal{P}(A_\alpha) = (\mathcal{P}(A))_\alpha.$$

Proposition 3. Let A_1, A_2 be fuzzy multisets of X. Then,

$$
\begin{aligned}
f[A_1 \oplus A_2] &= f[A_1] \oplus f[A_2] && (7)\\
f[A_1 \cup A_2] &\supseteq f[A_1] \cup f[A_2] && (8)\\
f[A_1 \cap A_2] &\subseteq f[A_1] \cap f[A_2] && (9)\\
f(A_1 \oplus A_2) &\subseteq f(A_1) \oplus f(A_2) && (10)\\
f(A_1 \cup A_2) &= f(A_1) \cup f(A_2) && (11)\\
f(A_1 \cap A_2) &\subseteq f(A_1) \cap f(A_2) && (12)\\
\mathcal{P}(A_1 \oplus A_2) &= \mathcal{P}(A_1) \cup \mathcal{P}(A_2) && (13)\\
\mathcal{P}(A_1 \cup A_2) &= \mathcal{P}(A_1) \cup \mathcal{P}(A_2) && (14)\\
\mathcal{P}(A_1 \cap A_2) &= \mathcal{P}(A_1) \cap \mathcal{P}(A_2) && (15)
\end{aligned}
$$

3 Multivariable functions

Let us consider three universes $W = \{w_1, ..., w_\ell\}$, $X = \{x_1, ..., x_m\}$ and $Y = \{y_1, ..., y_n\}$. Consider two-variable function $f: W \times X \to Y$. Our purpose is to consider $f[A, B]$ where A (resp. B) is a fuzzy multiset of W (resp. X).

Assume

$$
\begin{aligned}
A &= \{(w, \mu), ..., (w', \mu')\}\\
B &= \{(x, \nu), ..., (x', \nu')\}
\end{aligned}
$$

in which a symbol may be repeated. A multiset Cartesian product is defined:

$$A \times B = \{((w, x), \mu \wedge \nu), ..., ((w', x'), \mu' \wedge \nu')\}$$

in which all combinations of symbols in A and those in B are listed.

Now, the definition of $f[A, B]$ is as follows.

$$f[A, B] = \{(f(w, x), \mu \wedge \nu), ..., (f(w', x'), \mu' \wedge \nu')\} \tag{16}$$

Then we have

Proposition 4.

$$f(\mathcal{P}(A), \mathcal{P}(B)) = \mathcal{P}(f[A, B]).$$

Proposition 5. For an arbitrary $\alpha \in (0, 1]$,

$$f[A_\alpha, B_\alpha] = (f[A, B])_\alpha$$

Proposition 6. Let A_1, A_2 be fuzzy multisets of W and B be a fuzzy multiset of X. Then,

$$f[A_1 \oplus A_2, B] = f[A_1, B] \oplus f[A_2, B] \quad (17)$$
$$f[A_1 \cup A_2, B] \supseteq f[A_1, B] \cup f[A_2, B] \quad (18)$$
$$f[A_1 \cap A_2, B] \subseteq f[A_1, B] \cap f[A_2, B] \quad (19)$$
$$f(A_1 \oplus A_2, B) \subseteq f(A_1, B) \oplus f(A_2, B) \quad (20)$$
$$f(A_1 \cup A_2, B) = f(A_1, B) \cup f(A_2, B) \quad (21)$$
$$f(A_1 \cap A_2, B) \subseteq f(A_1, B) \cap f(A_2, B) \quad (22)$$

Functions $f(w, x, ..., z)$ of many variables can be dealt with in the same way. We omit the detail.

4 Conclusion

A multiset-valued image of fuzzy multisets has been studied and theoretical properties have been contrasted with the ordinary image. In particular, commutativity of the new image with α-cut has been proved.

Applications include query languages for fuzzy database, since it is well-known that the ordinary SQL handles crisp multisets. Therefore fuzzy SQL [1] should deal with fuzzy multisets. Moreover applications to information retrieval [7] should be studied.

Another interesting property is that infinite fuzzy multisets should be studied [9], while all arguments for the crisp case is finite. Thus, future studies of the theory and applications are promising.

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Appendix A: Basic operations of fuzzy multisets

A crisp multiset of X is characterized by $C_M: X \to$ N which implies that the number of copies of x is $C_M(x)$ We may write a multiset M as $M = \{k_1/x_1, ..., k_n/x_n\}$ or $M = \{\overbrace{x_1, .., x_1}^{k_1}, ..., \overbrace{x_n, .., x_n}^{k_n}\}$ to show $C_M(x_i) = k_i$.

For example, let $X = \{a, b, c, d\}$. A multiset

$$M = \{a, a, b, b, b, c\}$$

is expressed as

$$M = \{2/a, 3/b, 1/c, 0/d\}.$$

We have $C_M(a) = 3$, $C_M(b) = 2$, $C_M(c) = 1$, $C_M(d) = 0$.

Basic relations and operations for crisp multisets are ass follows.

- (inclusion):
  $M \subseteq N \Leftrightarrow C_M(x) \leq C_N(x), \forall x \in X.$
- (equality):
  $M = N \Leftrightarrow C_M(x) = C_N(x), \forall x \in X.$
- (union):
  $C_{M \cup N}(x) = \max[C_M(x), C_N(x)].$
- (intersection):
  $C_{M \cap N}(x) = \min[C_M(x), C_N(x)].$
- (addition):
  $C_{M \oplus N}(x) = C_M(x) + C_N(x).$

A fuzzy multiset is defined to be a crisp multiset of $X \times [0, 1]$ [13]. An example is

$$A = \{(a, 0.2), (a, 0.3), (b, 1), (b, 0.5), (b, 0.5)\}$$

of the universe $X = \{a, b, c, d\}$.

For an arbitrary $x \in X$, we can arrange those (x, μ) into the decreasing order of thee membership. We thus write

$$\mu_A^1(x), \mu_A^2(x), ..., \mu_A^p(x)$$

$(\mu_A^1(x) \geq \mu_A^2(x) \geq ... \geq \mu_A^p(x))$ and call this sequence a membership sequence.

If the universe is finite, we can assume that all the membership sequences are of the same length, by appending appropriate numbers of zero memberships.

For the above example,

$$A = \{(0.3, 0.2, 0)/a, (1, 0.5, 0.5)/b, (0, 0, 0)/c, (0, 0, 0)/d\}.$$

1. **inclusion:**
   $A \subseteq B \Leftrightarrow \mu_A^j(x) \leq \mu_B^j(x),\ j = 1, .., p,\ \forall x \in X.$
2. **equality:**
   $A = B \Leftrightarrow \mu_A^j(x) = \mu_B^j(x),\ j = 1, .., p,\ \forall x \in X.$
3. **union:**
   $\mu_{A \cup B}^j(x) = \mu_A^j(x) \vee \mu_B^j(x),\ j = 1, .., p,\ \forall x \in X.$
4. **intersection:**
   $\mu_{A \cap B}^j(x) = \mu_A^j(x) \wedge \mu_B^j(x),\ j = 1, .., p,\ \forall x \in X.$
5. **α-cut:**
   $\mu_A^1(x) < \alpha \Rightarrow C_{A_\alpha}(x) = 0,$
   $\mu_A^j(x) \geq \alpha,\ \mu_A^{j+1}(x) < \alpha \Rightarrow C_{A_\alpha}(x) = j,$
   $j = 1, ..., p.$

6. **addition:** Addition is defined by the addition of crisp multisets of $X \times [0,1]$.

Proposition A1. Let A and B are fuzzy multisets of X. Then, $A \subseteq B$ if and only if $A_\alpha \subseteq B_\alpha$ for every $\alpha \in (0,1]$. Moreover $A = B$ if and only if $A_\alpha = B_\alpha$ for every $\alpha \in (0,1]$.

Proposition A2. Let A and B are fuzzy multisets of X, and fix $\alpha \in (0,1]$ arbitrarily.

$$\begin{aligned}(A \cup B)_\alpha &= A_\alpha \cup B_\alpha, \\ (A \cap B)_\alpha &= A_\alpha \cap B_\alpha.\end{aligned}$$

Proposition A3. Let A, B, and C be fuzzy multisets of X. Then we have

$$A \cup B = B \cup A, \qquad A \cap B = B \cap A,$$

$$\begin{aligned}A \cup (B \cup C) &= (A \cup B) \cup C, \\ A \cap (B \cap C) &= (A \cap B) \cap C, \\ (A \cap B) \cup C &= (A \cup C) \cap (B \cup C), \\ (A \cup B) \cap C &= (A \cap C) \cup (B \cap C).\end{aligned}$$

Appendix B: Proofs of propositions

Proof of Proposition 1

If $y \notin f(X)$,

$$\mu_{f(\mathcal{P}(A))}(y) = \mu_{\mathcal{P}(f[A])}(y) = 0.$$

If $y \in f(X)$,

$$\begin{aligned}\mu_{f(\mathcal{P}(A))}(y) &= \bigvee_{x \in f^{-1}(y)} \mu_{\mathcal{P}(A)}(x) \\ &= \bigvee_{x \in f^{-1}(y)} \mu_A^1(x) = \mu_{\mathcal{P}(f[A])}(y).\end{aligned}$$

Proof of Proposition 2

Let us fix an arbitrary $\alpha \in (0,1]$. Assume also that $A = \{(x', \mu'), ..., (x'', \mu'')\}$. Suppose that $\{(z, \nu), ..., (z'', \nu'')\}$ is the collection of all elements in A such that $\mu \geq \alpha$. Then, $A_\alpha = \{(z,1), ..., (z'',1)\}$, hence

$$f[A_\alpha] = \{(f(z),1), ..., (f(z''),1)\} = (f[A])_\alpha.$$

For $\mathcal{P}$,

$$x \in \mathcal{P}(A_\alpha) \Leftrightarrow \mu_A^1(x) \geq \alpha \Leftrightarrow x \in \mathcal{P}(A)_\alpha.$$

Proof of Proposition 3

Let us first assume that all multisets are crisp. For addition, $A_1 \oplus A_2$ simply lists elements in A_1 and A_2 with redundancies. Hence $f[A_1 \oplus A_2]$ is exactly the same as thee listing of $f[A_1]$ and $f[A_2]$. It therefore is obvious that

$$f[A_1 \oplus A_2] = f[A_1] \oplus f[A_2].$$

Next, notice

$$C_{f[A_1]}(x) = \sum_{x \in f^{-1}(y)} C_{A_1}(x),$$

whereby we have

$$\begin{aligned}C_{f[A_1] \cup f[A_2]}(y) &= C_{f[A_1]}(y) \vee C_{f[A_2]}(y) \\ &= \Big(\sum_{x \in f^{-1}(y)} C_{A_1}(x)\Big) \\ &\quad \vee \Big(\sum_{x \in f^{-1}(y)} C_{A_2}(x)\Big) \\ &\leq \sum_{x \in f^{-1}(y)} (C_{A_1}(x) \vee C_{A_2}(x)) \\ &= C_{f[A_1 \cup A_2]}(y); \\ C_{f[A_1] \cap f[A_2]}(y) &= C_{f[A_1]}(y) \wedge C_{f[A_2]}(y) \\ &= \Big(\sum_{x \in f^{-1}(y)} C_{A_1}(x)\Big) \\ &\quad \wedge \Big(\sum_{x \in f^{-1}(y)} C_{A_2}(x)\Big) \\ &\geq \sum_{x \in f^{-1}(y)} (C_{A_1}(x) \wedge C_{A_2}(x)) \\ &= C_{f[A_1 \cap A_2]}(y).\end{aligned}$$

The other relations are proved likewise.

Now, consider fuzzy multisets. For proving the same set of relations for the fuzzy case, we can use Proposition 2 and Proposition A2. For example,

$$\begin{aligned}(f[A_1] \cup f[A_2])_\alpha &= f[A_1]_\alpha \cup f[A_2]_\alpha \\ &= f[A_{1\alpha}] \cup f[A_{2\alpha}] \\ &\supseteq f[A_{1\alpha} \cup A_{2\alpha}] \\ &= f[A_1 \cup A_{2\alpha}] \\ &= f[A_1 \cup A_2]_\alpha\end{aligned}$$

From Proposition A1, we have thee second relation. The other relations are proved in the same way.

Proof of Proposition 4

Let (w_1, μ_1) and (x_1, ν_1) be the elements for which μ_1 (resp.ν_1) is the maximum among all μ's (resp. ν's). Then the left hand side is $(f(w_1, x_1), \mu_1 \wedge \nu_1)$ using the extension principle. while the element of the maximum membership in $f[A, B]$ is obviously $\mu_1 \wedge \nu_1$ with the corresponding element is $f(w_1, x_1)$. Thus the desired equality follows.

Proof of Proposition 5

The proof is almost the same as that for Proposition 2, and therefore the details are omitted.

Proof of Proposition 6

As the proof of Proposition 3, let us first suppose that all multisets are crisp.

The first equation is obvious. See the proof of Proposition 3.

For the second relation, we observe

$$\begin{aligned}
& C_{f[A_1,B]\cup f[A_2,B]}(y) \\
= \; & C_{f[A_1,B]}(y) \vee C_{f[A_2,B]}(y) \\
= \; & (\sum_{(w,x)\in f^{-1}(y)} C_{A_1\times B}(w,x)) \\
& \vee(\sum_{(w,x)\in f^{-1}(y)} C_{A_2\times B}(w,x)) \\
\leq \; & \sum_{(w,x)\in f^{-1}(y)} (C_{A_1\times B}(w,x) \\
& \vee C_{A_2\times B}(w,x)) \\
= \; & \sum_{(w,x)\in f^{-1}(y)} C_{A_1}(w)\cdot C_B(x) \\
& \vee C_{A_2}(w)\cdot C_B(w,x)) \\
= \; & \sum_{(w,x)\in f^{-1}(y)} (C_{A_1}(w)\vee C_{A_2}(w))\cdot C_B(w,x)) \\
= \; & \sum_{(w,x)\in f^{-1}(y)} (C_{(A_1\cup A_2)\times B}(w,x) \\
= \; & C_{f[A_1\cup A_2,B]}(y).
\end{aligned}$$

The third relation is proved as follows.

$$\begin{aligned}
& C_{f[A_1,B]\cap f[A_2,B]}(y) \\
= \; & C_{f[A_1,B]}(y) \wedge C_{f[A_2,B]}(y) \\
= \; & (\sum_{(w,x)\in f^{-1}(y)} C_{A_1\times B}(w,x)) \\
& \wedge(\sum_{(w,x)\in f^{-1}(y)} C_{A_2\times B}(w,x)) \\
\geq \; & \sum_{(w,x)\in f^{-1}(y)} (C_{A_1\times B}(w,x) \\
& \wedge C_{A_2\times B}(w,x)) \\
= \; & \sum_{(w,x)\in f^{-1}(y)} C_{A_1}(w)\cdot C_B(x) \\
& \wedge C_{A_2}(w)\cdot C_B(w,x)) \\
= \; & \sum_{(w,x)\in f^{-1}(y)} (C_{A_1}(w)\wedge C_{A_2}(w))\cdot C_B(w,x)) \\
= \; & \sum_{(w,x)\in f^{-1}(y)} (C_{(A_1\cap A_2)\times B}(w,x) \\
= \; & C_{f[A_1\cap A_2,B]}(y).
\end{aligned}$$

In the fourth relation, the right side is a multiset, while the left side is an ordinary set, and hence the inclusion is valid. For thee rest two relations, the situation is just the same as those for ordinary sets in view of the definition $f(\cdot)$. Thus all relations for the crisp case is proved.

For the fuzzy case, we can use the commutativity of the operations with the α-cut, as in the proof of Proposition 3. For example, using Proposition 5, we have

$$\begin{aligned}
& (f[A_1\cup A_2,B])_\alpha \\
= \; & (f[(A_1\cup A_2)_\alpha, B_\alpha] \\
= \; & (f[(A_1)_\alpha \cup (A_2)_\alpha, B_\alpha] \\
\supseteq \; & (f[(A_1)_\alpha, B_\alpha] \cup f[(A_2)_\alpha, B_\alpha] \\
= \; & f[A_1,B]_\alpha \cup f[A_2,B]_\alpha \\
= \; & (f[A_1,B]\cup f[A_2,B])_\alpha.
\end{aligned}$$

From Proposition A1, we have the second relation for the fuzzy case. The other relations are proved in exactly the same way.

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