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Extended Fuzzy DEA

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Abstract: DEA (data envelopment analysis) is a non-parametric technique for measuring and evaluating the relative efficiencies of a set of entities with common crisp inputs and outputs. In fact, in a real evaluation problem input and output data of entities often fluctuate. These fluctuating data can be represented as linguistic variables characterized by fuzzy numbers. Based on a fundamental CCR model, a fuzzy DEA model is proposed to deal with fuzzy input and output data. Furthermore, a model that extends a fuzzy DEA to a more general case is also proposed with considering the relation between DEA and RA (regression analysis). The crisp efficiency in CCR model is extended to an L-R fuzzy number in fuzzy DEA problems to reflect some uncertainty in real evaluation problems.

Keywords: Fuzzy sets, Data envelopment analysis (DEA), Regression analysis (RA).

1. Introduction

Data envelopment analysis (DEA) developed by Charnes et al [2,3,4,6,7] is a non-parametric technique for measuring and evaluating the relative efficiencies of a set of entities with the common inputs and outputs. Examples include school, hospital, libraries and, more recently, whole economic and society systems, in which outputs and inputs are always multiple in character. Most of papers make an assumption that input and output data are crisp ones without any variation. In fact, inputs and outputs of DMUs are ever-changeful. For example, for evaluating airplane company efficiencies seat-kilometers available, cargo-kilometers available, fuel and labor are inputs and passenger-kilometers performed is an output[7]. It is common sense that inputs and outputs are easy to change because of weather, season, operating state and so on. Because DEA is a 'boundary' method sensitive to outliers, it is very difficult to evaluate the unit's efficiency with varying inputs and outputs by conventional DEA models. Some authors [8,11] have proposed a lot of models to challenge how to deal with the variation of data in efficiency evaluation problems by stochastic models. In a more general case, the data for evaluation are often collected from the investigation by polling where the natural language such as *good*, *medium* and *bad* are used to reflect a sort of general situation of the investigated entities rather than a special case [9,10]. For example, experts with a long time working experience can make such general conclusions that airline A's passenger-kilometer is about *200 passenger-kilometers* and fuel cost is *high*. These fuzzy concepts are used to summarize the fluctuating of inputs and outputs, where the center of a fuzzy number represents the most general case and the spread reflects some possibilities. Given fuzzy input and output data, evaluating the relative efficiencies of entities can be regarded as a kind of computing with words [14]. In this paper, considering fuzzy input and output data, a fuzzy DEA model is proposed, and the corresponding efficiency measures are also defined as fuzzy numbers. Furthermore a model that extends the fuzzy DEA to a more general form is also proposed with considering the relation between DEA and regression analysis (RA). Since the proposed fuzzy DEA models can deal with perceptual information, a perceptual evaluation can be done by the proposed methods. It can be concluded that CCR model is extended to the general model with fuzzy data.

2. DEA and its relation with RA

2.1 CCR (Charnes-Cooper-Rhodes) model:

CCR model is a linear programming (LP) based method developed by charnes et al [2]. In CCR model the efficiency of entity evaluated is obtained as a ratio of the weighted sum of outputs to that of inputs subject to the condition that the similar ratio for every entity is not larger than 1. Mathematically, it is described as follows:

$$\max_{\boldsymbol{\mu},\boldsymbol{\nu}} \quad \frac{\boldsymbol{\mu}^t \mathbf{y}_o}{\boldsymbol{\nu}^t \mathbf{x}_o}$$

$$\text{S.T.} \quad \frac{\boldsymbol{\mu}^t \mathbf{y}_j}{\boldsymbol{\nu}^t \mathbf{x}_j} \le 1, \ (j=1,\dots,n), \qquad (1)$$

$$\boldsymbol{\mu} \ge \mathbf{0}, \quad \boldsymbol{\nu} \ge \mathbf{0},$$

where the evaluated entities (j=1,...,n) called DMUs form a reference set, $\mathbf{y}_j=[y_{j1},y_{j2},\cdots,y_{jm}]^t$ and $\mathbf{x}_j=[x_{j1},x_{j2},\cdots,x_{js}]^t$ are the given positive output and input vectors of the jth DMU, respectively, m and s are the numbers of outputs and inputs of each DMU, respectively, n is the number of DMUs, $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ are the coefficient vectors of $\mathbf{y}_j$ and $\mathbf{x}_j$, respectively and the index o indicates the evaluated DMU. $\boldsymbol{\mu}\geq\mathbf{0}$ represents the vector whose elements are zero or positive values but at least one element is not zero whereas $\boldsymbol{\mu}>\mathbf{0}$ represents the vector whose elements are all positive values.

The model (1) is equivalent to the following LP problem.

$$\begin{aligned}
\max_{\mu,\nu}\quad & \boldsymbol{\mu}^t\mathbf{y}_o \\
\text{S.T.}\quad & \boldsymbol{\nu}^t\mathbf{x}_o=1, \\
& \boldsymbol{\mu}^t\mathbf{y}_j\leq\boldsymbol{\nu}^t\mathbf{x}_j,\ (j=1,\dots,n), \\
& \boldsymbol{\mu}\geq\mathbf{0}, \\
& \boldsymbol{\nu}\geq\mathbf{0}.
\end{aligned}\tag{2}$$

2.2 Relation with RA

In essence, the efficient input-output levels in DEA are those which are not dominated by any others in the reference set, while regression analysis (RA) is an average method in which it estimates an average level for dependent variables by explanatory variables. Many authors have contrasted the use of RA and DEA as methods for comparative performance assessments [1,4,5,13]. CCR model and RA method can be regarded as two special cases of the following goal programming.

$$\begin{aligned}
\min_{\rho_i,\eta_i,\mu,\nu}\quad & E=\sum_{i=1}^{n}(a_i\rho_i+b_i\eta_i) \\
\text{S.T.}\quad & \boldsymbol{\mu}^t\mathbf{y}_i-\boldsymbol{\nu}^t\mathbf{x}_i=\rho_i-\eta_i,\ (i=1,\dots,n), \\
& \boldsymbol{\nu}^t\mathbf{x}_o=1, \\
& \boldsymbol{\mu}\geq\mathbf{0}, \\
& \boldsymbol{\nu}\geq\mathbf{0}, \\
& \rho_i\geq 0, \\
& \eta_i\geq 0,
\end{aligned}\tag{3}$$

where ρ_i and η_i represent the positive and negative deviations between $\boldsymbol{\mu}^t\mathbf{y}_i$ and $\boldsymbol{\nu}^t\mathbf{x}_i$, respectively, a_i and b_i are the coefficients of ρ_i and η_i, respectively. For the case of $a_i=b_i=1.$, the model becomes LAV(the least absolute values) estimator which evaluate the efficiency from the average level, while for the case of $a_i\to+\infty$, $b_i\to 0\,(i\neq \mathrm{o}, b_o=1)$, the model becomes DEA (2) which evaluates the efficiency from the superior level. The associated efficiency measure is defined as

$$\boldsymbol{\mu}^t\mathbf{y}_o=\rho_o-\eta_o+\boldsymbol{\nu}^t\mathbf{x}_o=1+\rho_o-\eta_o,\tag{4}$$

3. Fuzzy DEA models

3.1 Fuzzy linear systems

A fuzzy set **A** defined on the n-dimensional space is called an n-dimensional fuzzy vector $\mathbf{A}=[A_1,\cdots,A_n]$, for simplicity, whose element A_j is characterized by a triangular membership function as follows.

$$\mu_{A_j}(x_j)=\begin{cases}1-|x_j-a_j|/c_j & a_j-c_j\leq x_j\leq a_j+c_j, c_j>0\\ 0 & otherwise\end{cases},\tag{5}$$

where a_j and c_j are the center and the spread of A_j, respectively. The membership function of the fuzzy vector **A** is defined as follows.

$$\mu_{\mathbf{A}}(\mathbf{x})=\mu_{A_1}(x_1)\wedge\mu_{A_2}(x_2)\wedge\cdots\wedge\mu_{A_n}(x_n),\tag{6}$$

where $\mathbf{x}^t=[x_1,\cdots,x_n]$. **A** is simply denoted as $(\mathbf{a},\mathbf{c})$ with the center vector $\mathbf{a}^t=[a_1,\cdots,a_n]$ and the spread vector $\mathbf{c}^t=[c_1,\cdots,c_n]$.

Given a fuzzy vector $\mathbf{A}=(\mathbf{a},\mathbf{c})$, a fuzzy linear function

$$\mathbf{Y}=A_1x_1+\cdots+A_nx_n,\tag{7}$$

can be represented as a triangular fuzzy variable as follows[12]:

$$\mathbf{Y}=(\mathbf{x}^t\mathbf{a},|\mathbf{x}|^t\mathbf{c}),\tag{8}$$

where $\mathbf{x}^t=[x_1,\cdots,x_n]$.

3.2 Fuzzy DEA

Given fuzzy input and output data, CCR model (2) can be naturally extended to the following fuzzy DEA model.

$$\begin{aligned}
\max_{\mu\nu}\quad & \boldsymbol{\mu}^t\mathbf{Y}_o \\
\text{S.T.}\quad & \boldsymbol{\nu}^t\mathbf{X}_o\approx\tilde{1}, \\
& \boldsymbol{\mu}^t\mathbf{Y}_j\underset{\sim}{<}\boldsymbol{\nu}^t\mathbf{X}_j,\ (j=1,\dots,n), \\
& \boldsymbol{\mu}\geq\mathbf{0}, \\
& \boldsymbol{\nu}\geq\mathbf{0},
\end{aligned}\tag{9}$$

where $\mathbf{X}_j=(\mathbf{x}_j,\mathbf{c}_j)$ and $\mathbf{Y}_j=(\mathbf{y}_j,\mathbf{d}_j)$ are an s-dimensional fuzzy input vector and an m-dimensional fuzzy output vector of the jth DMU. It can be seen that fuzzy input and output vectors in (9) take the place of crisp input and output vectors in (2). As a result, the relations of "almost equal" and "almost larger than" respectively corresponding to "equal" and "larger than" and "maximizing a fuzzy number" corresponding to "maximizing crisp output" are introduced in (9). Moreover, a fuzzy number $\tilde{1}=(1,e)$ replaces the real number 1 in (2) where $e<1$ is the predefined spread of fuzzy number $\tilde{1}$. Here $\mathbf{x}_j-\mathbf{c}_j>\mathbf{0}$ and $\mathbf{y}_j-\mathbf{d}_j>\mathbf{0}$ are assumed because we consider positive fuzzy numbers.

In what follows, let us consider how to explain fuzzy inequality $\boldsymbol{\mu}^t\mathbf{Y}_j\underset{\sim}{<}\boldsymbol{\nu}^t\mathbf{X}_j$, maximizing a fuzzy number $\boldsymbol{\mu}^t\mathbf{Y}_o$ and fuzzy equal $\boldsymbol{\nu}^t\mathbf{X}_o\approx\tilde{1}$ in (9).

Definition 1: Given two fuzzy variables $Z_1=(z_1,w_1)$ and $Z_2=(z_2,w_2)$ characterized by triangular membership functions, the fuzzy inequality $Z_1\underset{\sim}{<}Z_2$ can be defined by the following inequalities.

$$z_1-(1-h)w_1\leq z_2-(1-h)w_2,\tag{10}$$

$$z_1 + (1-h)w_1 \le z_2 + (1-h)w_2, \quad (11)$$

where $0 \le h \le 1$ is a predefined value of the fuzzy membership function. It is clear that the fuzzy "$\tilde{<}$" is defined by comparison of the endpoints of fuzzy number Z_1 and Z_2.

After defining a fuzzy inequality, let us consider maximizing a fuzzy number. Keeping the consistency with the concept of fuzzy inequality defined above, "Maximizing a fuzzy number Z=(z, w)" can be explained as simultaneously maximizing $z-(1-h)w$ and $z+(1-h)w$, which can be dealt with by maximizing a weighted function $\lambda_1(z-(1-h)w)+\lambda_2(z+(1-h)w)$, where $\lambda_1 \ge 0$, $\lambda_2 \ge 0$ and $\lambda_1 + \lambda_2 = 1$. If $\lambda_1 = 1$, it is regarded as a pessimistic opinion of maximizing fuzzy number Z because the worst situation is considered whereas if $\lambda_2 = 1$, it is regarded as an optimistic opinion of maximizing fuzzy number Z because the best situation is considered. In this paper, the case of $\lambda_1 = 1$ is considered, that is,

$$\max \quad z-(1-h)w. \quad (12)$$

Next, let us consider how to obtain ν such that $\nu^t \mathbf{X}_o \approx \tilde{1} =(1,e)$ in (9) where $\nu^t \mathbf{X}_o \approx \tilde{1}$ plays a role of the normalized pseudo-input as in CCR model (2). An explanation of $\nu^t \mathbf{X}_o \approx \tilde{1}$ can be deduced from **Definition 1**. That is, $\nu^t \mathbf{X}_o \approx \tilde{1}$ is regarded as a kind of limit case under the definition of $\nu^t \mathbf{X}_o \tilde{\le} \tilde{1}$. In other words, the fuzzy number $\nu^t \mathbf{X}_o$ should move rightwards as much as possible. In the limit case, the left endpoints of $\nu^t \mathbf{X}_o$ and $\tilde{1}$ overlap while the right endpoint of $\nu^t \mathbf{X}_o$ is not larger than that of $\tilde{1}$. Thus, the problem for finding ν such that $\nu^t \mathbf{X}_o \approx \tilde{1}$ can be converted into the following optimization.

$$\begin{aligned} \max_{\nu} \quad & \nu^t \mathbf{c}_o \\ \text{S.T.} \quad & \nu^t \mathbf{x}_o - (1-h)\nu^t \mathbf{c}_o = 1-(1-h)e, \\ & \nu^t \mathbf{x}_o + (1-h)\nu^t \mathbf{c}_o \le 1+(1-h)e, \quad (13) \\ & \nu \ge \mathbf{0}. \end{aligned}$$

It can be seen that (13) is used to find out a fuzzy number $Z = \nu^t \mathbf{X}_o$ with the largest spread and the same left endpoint as fuzzy number $\tilde{1}$ in h-level set.

Using formulations (10), (11), (12) and (13), a crisp version of (9) is obtained as follows:

$$\begin{aligned} \max_{\mu\nu} \quad & \mu^t \mathbf{y}_o - (1-h)\mu^t \mathbf{d}_o \\ \text{S. T.} \quad & \max_{\nu} \ \nu^t \mathbf{c}_o \\ & \quad \text{S.T.} \quad \nu^t \mathbf{x}_o - (1-h)\nu^t \mathbf{c}_o = 1-(1-h)e, \\ & \qquad\qquad \nu^t \mathbf{x}_o + (1-h)\nu^t \mathbf{c}_o \le 1+(1-h)e, \\ & \qquad\qquad \nu \ge \mathbf{0}, \\ & \mu^t \mathbf{y}_j - (1-h)\mu^t \mathbf{d}_j \le \nu^t \mathbf{x}_j - (1-h)\nu^t \mathbf{c}_j \\ & \mu^t \mathbf{y}_j + (1-h)\mu^t \mathbf{d}_j \le \nu^t \mathbf{x}_j + (1-h)\nu^t \mathbf{c}_j, \ (j=1,\dots,n), \\ & \mu \ge \mathbf{0}. \end{aligned} \quad (14)$$

It should be noted that the optimization problem (13) is embedded into (14) to obtain the ν such that $\nu^t \mathbf{X}_o \approx \tilde{1}$. It can be seen that when $\mathbf{c}_i = \mathbf{0}$, $\mathbf{d}_i = \mathbf{0}$ and e=0, fuzzy DEA (14) just becomes CCR model (2). It means that the model (14) can evaluates the efficiencies of DMUs in more general mode, by which the crisp, fuzzy and hybrid inputs and outputs can be handled.

Theorem 1. If there is a feasible region in the constraints of (13), there exists an optimal solution in (14).

Theorem 2. If $\max[c_{o1}/x_{o1}, \cdots, c_{os}/x_{os}] < e$ in (13), the optimization solution ν restricted by $\nu \ge 0$ always exists.

Considering n DMUs, e is taken as $e = \max_{j=1,\dots,n}(\max_{k=1,\dots,s} c_{jk}/x_{jk})$ in the optimization problem (14).

The optimization problem (14) is equivalence to the following LP problem.

$$\begin{aligned} \max_{\mu\nu} \quad & \mu^t \mathbf{y}_o - (1-h)\mu^t \mathbf{d}_o \\ \text{S. T.} \quad & \nu^t \mathbf{c}_o \ge g_o \\ & \nu^t \mathbf{x}_o - (1-h)\nu^t \mathbf{c}_o = 1-(1-h)e, \\ & \nu^t \mathbf{x}_o + (1-h)\nu^t \mathbf{c}_o \le 1+(1-h)e, \\ & \mu^t \mathbf{y}_j - (1-h)\mu^t \mathbf{d}_j \le \nu^t \mathbf{x}_j - (1-h)\nu^t \mathbf{c}_j \quad (15) \\ & \mu^t \mathbf{y}_j + (1-h)\mu^t \mathbf{d}_j \le \nu^t \mathbf{x}_j + (1-h)\nu^t \mathbf{c}_j, \ (j=1,\dots,n) \\ & \mu \ge \mathbf{0}, \ \nu \ge \mathbf{0}, \end{aligned}$$

where g_o is the optimal value of the objective function of (13).

Definition 2: The fuzzy efficiency of the DMU_o with fuzzy input $\mathbf{X}_o=(\mathbf{x}_o, \mathbf{c}_o)$ and output vectors $\mathbf{Y}_0=(\mathbf{y}_o, \mathbf{d}_o)$ is defined as an L-R fuzzy number $\varepsilon = (w_l, \eta, w_r)$ as follows:

$$\begin{aligned} \eta &= \frac{\mu^{*t}\mathbf{y}_o}{\nu^{*t}\mathbf{x}_o}, \\ w_l &= \eta - \frac{\mu^{*t}(\mathbf{y}_o - \mathbf{d}_o(1-h))}{\nu^{*t}(\mathbf{x}_o + \mathbf{c}_o(1-h))}, \quad (16) \\ w_r &= \frac{\mu^{*t}(\mathbf{y}_o + \mathbf{d}_o(1-h))}{\nu^{*t}(\mathbf{x}_o - \mathbf{c}_o(1-h))} - \eta, \end{aligned}$$

where ν^* and μ^* are the obtained coefficient vectors from (15), w_l, w_r and η are the left, right spreads and center value of the fuzzy efficiency ε, respectively.

Definition 3. The DUM with $\eta + w_r \geq 1$ for *h*-level is called *h*-possibilistic D efficient DMU (*PD* DMU), on the contrary, the one with $\eta + w_r < 1$ for *h*-level is called *h*-possibilistic D inefficient DMU (*PDI* DMU). The set of all *PD* DMUs is called *h*-possibilistic nondominated set, denoted as S_h.

It is obvious that for *h*=1, *h*-possibilistic D efficient DMUs (*PD* DMUs) and *h*-possibilistic D inefficient DMUs (*PDI* DMUs) become the conventional D efficient DMUs and D inefficient DMUs.

Theorem 3. The center value of the fuzzy efficiency of any DMU obtained from (15) is not more than 1.

Theorem 3 implies that evaluating fuzzy efficiencies of DMUs by the model (15) is similar to evaluating crisp efficiencies of DMUs by CCR model (2) which seeks the nondominated one by other DMUs.

Now, we will discuss the given threshold *h*. If we take a large value for *h*, it means that we consider a relatively narrow range where all of the data have high possibilities. Conversely, if we take a small value for *h*, it means that we investigate the data in relatively wide range. For example, *h*=1 means that only the centers of fuzzy data need to be considered, while *h*=0 means that all of the possible data need to be considered.

3.3 Fuzzy model based on the model (3)

Let us consider an extension of the model (3) corresponding to fuzzy input and output data.

Using the formulations (7) and (8), $\mu^t \mathbf{y}_i$ and $\nu^t \mathbf{x}_i$ in (3) become the following two fuzzy set Z_{i0} and Z_{i1} associated with the fuzzy output and input vectors $\mathbf{Y}_i$ and $\mathbf{X}_i$, respectively.

$$Z_{i0} = \mu^t \mathbf{Y}_i, \tag{17}$$

$$Z_{i1} = \nu^t \mathbf{X}_i. \tag{18}$$

Definition 4: Given two fuzzy number $Z_1 = (z_1, e_1)$ and $Z_2 = (z_2, e_2)$, we define θ and ψ as:

$$\theta = z_1 - (1-h)e_1 - z_2 + (1-h)e_2, \tag{19}$$

$$\psi = z_1 + (1-h)e_1 - z_2 - (1-h)e_2, \tag{20}$$

where θ and ψ are differences between the left and right endpoints of $[Z_1]_h$ and $[Z_2]_h$, respectively. θ and ψ are called left and right deviations between Z_1 and Z_2.

With the aid of **Definition 4**, the model (3) is extended to the following model.

$$\min_{\theta_i,\psi_i,\mu,\nu} E = \sum_{j=1}^{n} \delta_j\theta_j + \delta_j^{'}\psi_j + \varphi_j\theta_j^{'} + \varphi_j^{'}\psi_j^{'}$$

$$\begin{aligned}
\text{S.T. } & \max_{\nu} \nu^t \mathbf{c}_o \\
& \quad \text{S.T } \nu^t \mathbf{x}_o - (1-h)\nu^t \mathbf{c}_o = 1 - (1-h)e, \\
& \quad \nu^t \mathbf{x}_o + (1-h)\nu^t \mathbf{c}_o \leq 1 + (1-h)e, \\
& \quad \nu \geq \mathbf{0}, \\
& \mu^t \mathbf{y}_j - (1-h)\mu^t \mathbf{d}_j - \nu^t \mathbf{x}_j + (1-h)\nu^t \mathbf{c}_j = \theta_j - \theta_j^{'}, \\
& \mu^t \mathbf{y}_j + (1-h)\mu^t \mathbf{d}_j - \nu^t \mathbf{x}_j - (1-h)\nu^t \mathbf{c}_j = \psi_j - \psi_j^{'}, \\
& \mu \geq \mathbf{0}, \\
& \theta_j \geq 0, \\
& \theta_j^{'} \geq 0, \\
& \psi_j \geq 0, \\
& \psi_j^{'} \geq 0,
\end{aligned} \tag{21}$$

where θ_j and $\theta_j^{'}$ (*j*=1,...,*n*) are positive and negative left-deviations between $\mu^t \mathbf{Y}_j$ and $\nu^t \mathbf{X}_j$, respectively, ψ_j and $\psi_j^{'}$ are positive and negative right-deviations between $\mu^t \mathbf{Y}_j$ and $\nu^t \mathbf{X}_j$, respectively. δ_j, $\delta_j^{'}$, φ_j, and $\varphi_j^{'}$ are the nonnegative coefficients of θ_j, Ψ_j, $\theta_j^{'}$ and $\psi_j^{'}$ for the *j*th DMU, respectively. As in the model (3), changing these coefficients can influence the relation between $\mu^t \mathbf{Y}_j$ and $\nu^t \mathbf{X}_j$, which means variation of evaluation standards. Speaking in detail, when $\delta_j \to \infty$, $\delta_j^{'} \to \infty$, $\varphi_j \to 0$ (*j*=1,...,*n*, $j \neq$ o, $\varphi_o = 1$) and $\varphi_j^{'} \to 0$ (*j*=1,...,*n*,), the model (21) approaches (14) and when $\delta_j = \delta_j^{'} = \varphi_j = \varphi_j^{'} = 1$ (*j*=1,...,*n*), the model becomes FLAV (fuzzy least absolute values) estimator. It means the model (21) is a general case of models (2), (3) and (14). The fuzzy efficiency measure is the same as (17). It is obvious that **Theorem 3** can not hold in (21).

Similarly, the optimization problem (21) is equivalence to the following LP problem.

$$\min_{\theta_i,\psi_i,\mu,\nu} E = \sum_{j=1}^{n} \delta_j\theta_j + \delta_j^{'}\psi_j + \varphi_j\theta_j^{'} + \varphi_j^{'}\psi_j^{'}$$

$$\begin{aligned}
\text{S.T. } & \nu^t \mathbf{c}_o \geq g_o \\
& \nu^t \mathbf{x}_o - (1-h)\nu^t \mathbf{c}_o = 1 - (1-h)e, \\
& \nu^t \mathbf{x}_o + (1-h)\nu^t \mathbf{c}_o \leq 1 + (1-h)e, \\
& \mu^t \mathbf{y}_j - (1-h)\mu^t \mathbf{d}_j - \nu^t \mathbf{x}_j + (1-h)\nu^t \mathbf{c}_j = \theta_j - \theta_j^{'}, \\
& \mu^t \mathbf{y}_j + (1-h)\mu^t \mathbf{d}_j - \nu^t \mathbf{x}_j - (1-h)\nu^t \mathbf{c}_j = \psi_j - \psi_j^{'}, \\
& \mu \geq \mathbf{0}, \nu \geq \mathbf{0}, \theta_j \geq 0, \theta_j^{'} \geq 0, \psi_j \geq 0, \psi_j^{'} \geq 0.
\end{aligned} \tag{22}$$

4. Conclusion

In this paper, two kinds of fuzzy DEA models are proposed for evaluating the efficiencies of DMUs with fuzzy input and output data. The obtained efficiencies are also fuzzy numbers to reflect the inherent fuzziness in evaluation problems. It can be concluded that the proposed fuzzy DEA models extend CCR model to more general cases where crisp, fuzzy and hybrid data can be handled easily. Because uncertainty always exists in human thinking and judgement, fuzzy DEA models can play an important role for perceptual evaluation problems comprehensively existing in the real world.

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