The Third AFSS(1998). 376-379

Fuzzy maps

S. H. Chung

Department of Mathematics, Kyungwon University,
San65 Bokjung-Dong, Sujung-Gu, Sungnam, Kyunggi-Do, 461-701 Korea
TEL : (0342)750-5387, FAX : (0342)750-5389, e-mail : sehwach@unitel.co.kr

Abstract

We introduce the concept of a 'fuzzy' map between sets by modifying the concept of the extension principle introduced by Dubois and Prade in [1] and study their properties. Using these we generalize Goguen's and Zadeh's extension principles in [2] and [3].

Keywords: DP-fuzzifing map, fuzzy map, (α,β)-fuzzy morphism, G-fuzzy morphism

1. Fuzzy maps

The unit interval will be denoted by I and for a set X, I^X denotes the set of all fuzzy sets on X. For $\alpha \in I^X$, $S(\alpha) = \{x \in X: \alpha(x) > 0\}$.

Dubois and Prade's Extension Principle [1]: For two sets X and Y, a map $e: X \to I^Y$ is said to be a DP-fuzzifing map if there is a fuzzy relation $\mu: X \times Y \to I$ such that for all $(x, y) \in X \times Y$, $e(x)(y) = \mu(x,y)$.

Definition 1.1 Let X and Y be two sets. A DP-fuzzifing map $e: X \to I^Y$ is said to be a *fuzzy map* if for each $x \in X$, $S(e(x))$ is a singleton set or the empty set.

Remark 1) For a DP-fuzzifing map $e: X \to I^Y$, The following are equivalent:

i) e is a fuzzy map.

ii) there is a map $\mu: X \times Y \to I$ such that for each $(x, y) \in X \times Y$, $e(x)(y) = \mu(x, y)$ and $\mu(x, y) \wedge \mu(x, z) > 0$ implies $y = z$.

iii) $e(X) \subseteq \{y_t: y \in Y, t \in I\}$, where $y_t: Y \to I$ is a fuzzy point defined by

$$y_t(x) = \begin{cases} t & \text{if} \quad x = y \\ 0 & \text{if} \quad x \neq y \end{cases}.$$

2) For a fuzzy map $e: X \to I^Y$, let $D_e = \{x \in X: S(e(x)) \neq \emptyset\}$ and $R_e = \bigcup\{S(e(x)): x \in X\}$. Let $f_e: D_e \to R_e$ be defined by $f_e(x) = y$ iff $y \in S(e(x))$. Then f_e is an onto map. For a map $f: X \to Y$ and let $e: X \to I^Y$ be defined by

$$e_f(x)(y) = \begin{cases} 1 & \text{if} \quad f(x) = y \\ 0 & \text{if} \quad f(x) \neq y \end{cases}$$

Then e_f is a fuzzy map. Moreover $f_{e_f} = f$ and $e \leq e_{f_e} = 1$ on D_e.

Example Let R be the set of real numbers and let $e: R \to I^R$ be defined by

$$e(x)(y) = \begin{cases} (1+(x-10)^{-2})^{-1} & \text{if} \quad y = 10 \text{ and } 10 < x \\ 0 & \textit{otherwise} \end{cases}.$$

Then e is a fuzzy map.

Definition 1.2 A fuzzy map $e: X \to I^Y$ is said to be *equipotent* if there is a fuzzy map $e': Y \to I^Z$ such that $e(x)(y) = e'(y)(x)$ for each $(x, y) \in X \times Y$. In this case we say that X and Y are *fuzzy equipotent* under e and we write $e' = e^{-1}$ and $X \cong Y$.

Proposition 1.3 If a fuzzy map $e: X \to I^Y$ is equipotent, then $f_e: D_e \to R_e$ is one - one.

Proposition 1.4 Let $e: X \to I^Y$ and $e': Y \to I^Z$ be fuzzy maps. For each $(x, z) \in X \times Z$, let $(e' \circ e)(x)(z) = \bigvee_{y \in Y} (e(x)(y) \wedge e'(y)(z))$. Then $e' \circ e: X \to I^Z$ is a fuzzy map.

Definition 1.5 The fuzzy map $e' \circ e$ in the above proposition is said to be the *composition of e and e'* if $R_e \subseteq D_{e'}$.

Proposition 1.6 1) If $f: X \to Y$ and $g: Y \to Z$ are functions. Then $e_{g \cdot f} = e_g \circ e_f$.

2) If $e: X \to I^Y$ and $e': Y \to I^Z$ are fuzzy maps, then $f_{e' \cdot e} = f_{e'} \circ f_e$.

Proposition 1.7 Let $e: X \to I^Y$, $e': Y \to I^Z$ and $e'': Z \to I^W$ be fuzzy maps. Then $(e'' \circ e') \circ e = e'' \circ (e' \circ e)$.

Theorem 1.8 Let $e: X \to I^Y$ and $e': Y \to I^Z$ be symmetric fuzzy maps. Then $e' \circ e: X \to I^Z$ is a symmetric fuzzy map.

Corollary 1.9 $\cong$ is an equivalence relation on Set.

2. Fuzzy morphisms

Definition 2.1 Let α, β be a fuzzy set in X and Y and $e: X \to I^Y$ a fuzzy map. Then

1) e is said to be *stabile* if for each $x \in X$, $S_\alpha(e(x)) \neq \emptyset$.

2) e is said to be a *(α, β)-fuzzy morphism* (or simply fuzzy morphism) if $y \in S_\alpha(e(x))$ implies $\alpha(x) \leq \beta(y)$.

Remark Let $e: X \to I^Y$ be a symmetric fuzzy morphism and $e^{-1}: Y \to I^X$ a fuzzy morphism. If $\alpha(x) \vee \beta(y) \leq e(x)(y)$ then $\alpha(x) = \beta(y)$.

Goguen's Extension Principle [2]: For $\alpha \in I^X$ and $\beta \in I^Y$ a map $f: X \to Y$ is said to be a G-fuzzy morphism if $\alpha \leq f^{-1}(\beta)$.

Proposition 2.2 Let $f: X \to Y$ be a map and let α and β fuzzy set in X and Y respectively. Then f is a G-fuzzy morphism if and only if $e_f: X \to I^Y$ is a stabile fuzzy morphism.

Proposition 2.3 Let $e: X \to I^Y$ and $e': Y \to I^Z$ be fuzzy morphisms. Suppose $y \in S_\alpha(e(x))$ implies $z \in S_\beta(e'(y))$ for some $z \in Z$. Then $e' \circ e: X \to I^Z$ is a fuzzy morphism. In particular if e and e' are stabile fuzzy morphisms, $e' \circ e$ is a stabile fuzzy morphism.

Zadeh's Extension Principle [3]: A map $f: X \to Y$ induces two fuzzifing maps
$f: I^X \to I^Y$ and $f^{-1}: I^Y \to I^X$
which are defined by

$$f(\alpha)(y) = sup_{y=f(x)} \alpha(x) \text{ for all } \alpha \in I^X$$

and

$$f^{-1}(\beta)(x) = \beta(f(x)) \quad \text{for all } \beta \in I^Y.$$

A fuzzy map $e: X \to I^Y$ induces two maps $e: I^X \to I^Y$ and $e^{-1}: I^Y \to I^X$ which are defined by

$$e(\alpha)(y) = \begin{cases} \bigvee_{y \in S_\alpha(e(x))} \alpha(x) & \text{if } \{x: y \in S_\alpha(e(x))\} \neq \emptyset \\ 0 & \text{if } \{x: y \in S_\alpha(e(x))\} = \emptyset \end{cases} \quad \text{for all } \alpha \in I^X$$

and

$$e^{-1}(\beta)(x) = \begin{cases} \beta(y) \wedge e(x)(y) & \text{if } y \in S(e(x)) \\ 1 & \text{if } S(e(x)) = \emptyset \end{cases} \quad \text{for all } \beta \in I^Y.$$

Remark Let $e: X \to I^Y$ be a fuzzy map. If $0 < t \leq e(x)(y)$ then $e(x_t) = y_t$ and $e^{-1}(y_t) = t$. In particular, if e is symmetric and $0 < t \leq e(x)(y)$ then $e^{-1}(y_t) = x_t$.

Proposition 2.4 Let $e: X \to I^Y$ be a fuzzy map. Then one has the following:

(1) $e(\alpha) = 0$ iff $\alpha = 0$, where $S(\alpha) \subseteq \{x: S_\alpha(e(x)) \neq \emptyset\}$.

(2) if $\alpha_1 \leq \alpha_2$ then $e(\alpha_1) \leq e(\alpha_2)$, where $S_{\alpha_1}(e(x)) \subseteq S_{\alpha_2}(e(x))$ for all $x \in X$.

(3) $e(\bigvee_{j \in J} \alpha_j) \leq \bigvee_{j \in J} e(\alpha_j)$.

(4) $e(\bigwedge_{j \in J} \alpha_j) \leq \bigwedge_{j \in J} e(\alpha_j)$.

(5) if $\beta_1 \leq \beta_2$ then $e^{-1}(\beta_1) \leq e^{-1}(\beta_2)$.

(6) $e^{-1}(\bigvee_{j \in J} \beta_j) = \bigvee_{j \in J} e^{-1}(\beta_j)$.

(7) $e^{-1}(\bigwedge_{j \in J} \beta_j) = \bigwedge_{j \in J} e^{-1}(\beta_j)$.

(8) $\alpha \leq e^{-1}(e(\alpha))$, where $S(\alpha) \subseteq \{x: S_\alpha(e(x)) \neq \emptyset\}$.

(9) $e(e^{-1}(\beta)) \leq \beta$.

(10) if e is symmetric then $e(\bigvee_{j\in J}\alpha_j) = \bigvee_{j\in J}e(\alpha_j)$.

Theorem 2.5 Let X be a set and β_j a fuzzy set in a set Y_j ($j \in J$). Suppose $(e_j : X \rightarrow I^{Y_j})_{j \in J}$ is a source of fuzzy maps. Then there is a fuzzy set α in X satisfying the following:

1) for each $j \in J$, $e_j: X \rightarrow I^{Y_j}$ is a staible fuzzy morphism.

2) Let Z be a set and γ a fuzzy set in Z. If $e: Z \rightarrow I^X$ is a fuzzy map such that for each $j \in J$, $e_j \circ e$ is a (stabile) fuzzy morphism and $x \in S_\gamma(e(z))$ implies $\gamma(z) \leq e_j(x)(y_j)$ for some $y_j \in Y_j$, then e is a (stabile) fuzzy morphism.

Theorem 2.6 Let Y be a set and α_j a fuzzy set in a set X_j ($j \in J$). Suppose $e_j: X_j \rightarrow I^Y$ is a sink of (stabile) fuzzy maps. Then there is a fuzzy set α in set X satisfying the following:

1) for each $j \in J$, $e_j: X_j \rightarrow I^Y$ is a (stabile) fuzzy morphism.

2) Let Z be a set and γ a fuzzy set in Z. If $e: Y \rightarrow I^Z$ is a fuzzy map such that for each $j \in J$, $e \circ e_j: X_j \rightarrow I^Z$ is a (stabile) fuzzy morphism and $y \in S_{\alpha_j}(e_j(z))$ implies $\alpha_j(x_j) \leq e(x)(z)$ for some $z \in Z$, then e is a (stabile) fuzzy morphism.

References

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