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Limit Properties in the Fuzzy Real Line (II)

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ABSTRACT

We introduce the concepts of extending a fuzzily constrained function and a fuzzy extending of a real function by using usual limit and illustrate them.

1. Introduction

In [2], we had some limit properties in the fuzzy real line. In this paper, we introduce the concept of limit to a fuzzily constrained function, define that certain functions can be fuzzily constrained by using limit and give examples. Furthermore, we introduce the concept of limit to the fuzzy extension of a real function and obtain the 2nd and 3rd fuzzy extensions of the real function together examples.

2. Preliminaries

Throughout this paper, $\mathbb{R}$ denote the real line and $\tilde{P}(\mathbb{R})$ denote the fuzzy power set of $\mathbb{R}$. And the other notations will be used as in [4]. In particular, x_α denote the fuzzy point with the support x and the value α.

Definition 2.1([3])**(Fuzzily Contrained Function)** Let X and Y be two universes and f be an ordinary function from X to Y. Let $A \in \tilde{P}(X)$ and $B \in \tilde{P}(Y)$. The function f is said to *have a fuzzy domain* A *and a fuzzy range* B *if*

$$\mu_A(x) \leq \mu_B[f(x)] \text{ for all } x \in X.$$

Definition 2.2([5])(Fuzzy Extension of an Ordinary Function) Let f be an ordinary function from X to Y. By means of the extension principle, the image $f(A)$ of $A \in \tilde{P}(X)$, a fuzzy set in Y, is defined by

$$\mu_{f(A)}(y) = \begin{cases} \sup_{x \in f^{-1}(y)} \mu_A(x) & \text{if } f^{-1}(y) \neq \phi, \\ 0 & \text{if } f^{-1}(y) = \phi. \end{cases}$$

3. Usual Limits Applied to Fuzzy Sets in $\mathbb{R}$

Definition 3.1.(Limit in a Fuzzily Constrained Function) Let a real function $f: \mathbb{R} \rightarrow \mathbb{R}$ have a fuzzy domain $A \in \tilde{P}(\mathbb{R})$ and a fuzzy range $B \in \tilde{P}(\mathbb{R})$. If f has the limit l at $x = a$ and if

$$\alpha = \mu_A(a) \leq \mu_B(l) = \beta,$$

we say that f *has the limit* l_β *in* B *at* $x_\lambda = a_\alpha$ *in* A, denoted by

$$\lim_{x_\lambda \xrightarrow{A} a_\alpha} f(x_\lambda) = l_\beta \quad \text{or} \quad f(x_\lambda) \xrightarrow{B} l_\beta \quad \text{as} \quad x_\lambda \xrightarrow{A} a_\alpha.$$

Remark. We can consider a left limit and a right limit of f at $x = a$ in the definition 3.1.

Example 1. Consider the function $f: \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$f(x) = \begin{cases} x^2 - 1 / x - 1 & \text{for } x \neq 1, \\ 0 & \text{for } x = 1. \end{cases}$$

Let $\mu_A(x) = 1/3$ and $\mu_B(x) = 1/2$ for all $x \in \mathbb{R}$. Then, f has a fuzzy domain A and a fuzzy range B and f has the limit $2_{1/2}$ in B at $1_{1/3}$ in A.

Definition 3.2.(Extension of a Fuzzily Constrained Function) Let f be a function from $D \subset \mathbb{R}$ to $\mathbb{R}$ and let $A, B \in \tilde{P}(\mathbb{R})$. If f has the limit a' at each $a \in \mathbb{R} - D$ and if

$$\begin{cases} \mu_A(x) \leq \mu_B[f(x)] & \text{for } x \in D, \\ \mu_A(a) \leq \mu_B(a') & \text{for } a \in \mathbb{R} - D, \end{cases}$$

we say that f *has a fuzzy domain* A *and a fuzzy range* B.

Example 2. Consider the function $f(x) = (x^2 - 1)(x - 2) / (x - 1)(x - 2)$ and fuzzy sets A, B given in the example 1. Then, the domain of f is the set $\mathbb{R} - \{1, 2\}$ and f has the limits $2_{1/2}$ and $3_{1/2}$ in B at $1_{1/3}$ and $2_{1/3}$ in A, respectively. Thus, f has the fuzzy domain A and the fuzzy range B.

Using the usual limit, we obtain different fuzzy extensions of a real function from the fuzzy extension of it given in the definition 2.2.

Definition 3.3. (Several Fuzzy Extensions of a Real Function) Let a real function $f: \mathbb{R} \to \mathbb{R}$ and $A \in \tilde{P}(\mathbb{R})$ be given. For $y \in \mathbb{R}$, let S^y denote the subset of $\mathbb{R}$ given by

$$S^y = \{a \in \mathbb{R} \mid f(x) \to y \quad as \quad x \to a\}.$$

Then, the fuzzy set $f(A)$ in $\mathbb{R}$ can be characterized by three ways:

(Method 1). (Definition 2.2.).

$$\mu_{f(A)}(y) = \begin{cases} \sup_{x \in f^{-1}(y)} \mu_A(x) & \text{if } f^{-1}(y) \neq \phi, \\ 0 & \text{if } f^{-1}(y) = \phi. \end{cases}$$

(Method 2).

$$\mu_{f(A)}(y) = \begin{cases} \sup_{x \in f^{-1}(y)} \mu_A(x) & \text{if } f^{-1}(y) \neq \phi, \\ \sup_{x \in S^y} \mu_A(x) & \text{if } f^{-1}(y) = \phi \text{ and } S^y \neq \phi, \\ 0 & \text{if } f^{-1}(y) \cup S^y = \phi. \end{cases}$$

(Method 3).

$$\mu_{f(A)}(y) = \begin{cases} \sup_{x \in f^{-1}(y) \cup S^y} \mu_A(x) & \text{if } f^{-1}(y) \cup S^y \neq \phi, \\ 0 & \text{if } f^{-1}(y) \cup S^y = \phi. \end{cases}$$

The fuzzy sets $f(A)$ obtained by using methods 1, 2 and 3 will be called *the 1st, 2nd and 3rd C-Z(Choi-Zadeh) fuzzy extensions of* f, respectively and will be denoted by $f^{CZ1}(A)$, $f^{CZ2}(A)$ and $f^{CZ3}(A)$, respectively.

Since $A \subset B$ implies $A \cup B = B$, the following is obvious.

Proposition 3.4. If $S^y \subset f^{-1}(y)$, then

$$\mu_{f^{CZ1}(A)}(y) = \mu_{f^{CZ2}(A)}(y) = \mu_{f^{CZ3}(A)}(y).$$

Remark. The converse of the proposition is not true. Consider $f: \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 & \text{for } x \in \mathbb{R} - \{0, 1\}, \\ 2 & \text{for } x \in \{0, 1\}. \end{cases}$$

Let $A \in \tilde{P}(\mathbb{R})$ be characterized by

$$\mu_A(x) = \begin{cases} 1/2 & \text{for } x \in \mathbb{R} - \{0, 1\}, \\ 1/3 & \text{for } x \in \{0, 1\}. \end{cases}$$

Then, $f^{-1}(1/2) = \mathbb{R} - \{0, 1\}$ and $S^{1/2} = \mathbb{R}$. But

$$\mu_{f^{CZ1}(A)}(1/2) = \mu_{f^{CZ2}(A)}(1/2) = \mu_{f^{CZ3}(A)}(1/2) = 1/2.$$

Examlpe 3. Let the function f: $\mathbb{R} \rightarrow \mathbb{R}$ (see figure 1) be defined by

$$f(x) = \begin{cases} x^2 / |x| + 1 & \text{for } x \neq 0, \\ k(\neq 1) & \text{for } x = 0. \end{cases}$$

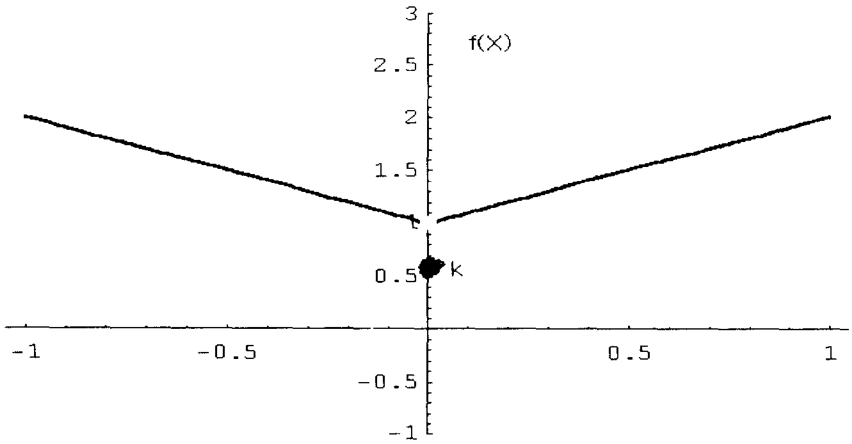

figure 1

Consider $A \in \tilde{P}(\mathbb{R})$ (see figure 2) characterized by

$$\mu_A(x) = \begin{cases} |x| / 2 + 1/2 & \text{for } |x| \leq 1, \\ 1 / |x| & \text{for } |x| \geq 1. \end{cases}$$

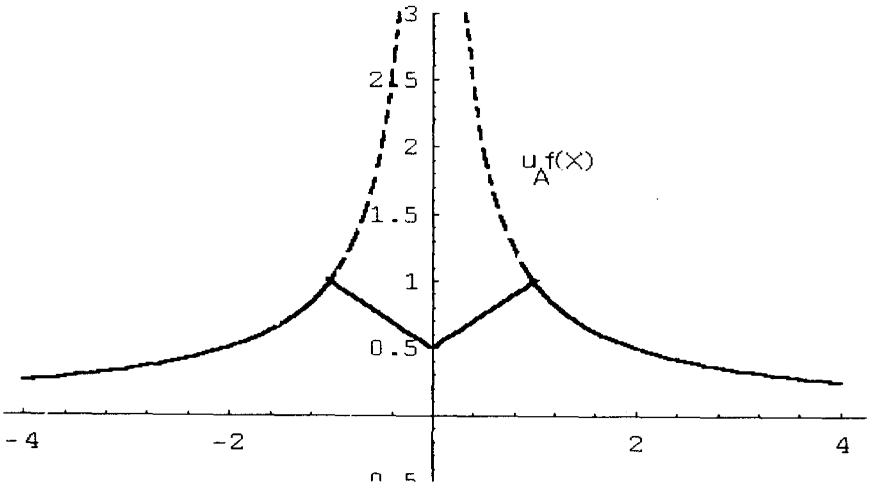

figure 2

Note that $f^{-1}(1) = \phi$ and $S^1 = \{0\}$. Thus, we have

$$\mu_{f^{(Z1)}(A)}(1) = 0, \quad \mu_{f^{(Z2)}(A)}(1) = \mu_{f^{(Z3)}(A)}(1) = \mu_A(0) = 1/2.$$

Note that for $k > 1$

$$f^{-1}(k) = \{1-k, 0, k-1\} \text{ and } S^k = \{1-k, k-1\}$$

and that for $k < 1$

$$f^{-1}(k) = \{0\} \quad \text{and} \quad S^k = \phi .$$

If $k > 3$, then

$$\mu_A(1-k) = \mu_A(k-1) = 1/(k-1) < 1/2 = \mu_A(0),$$

if $2 \le k \le 3$, then

$$\mu_A(1-k) = \mu_A(k-1) = 1/(k-1) \ge 1/2 = \mu_A(0),$$

and if $1 < k \le 3$, then

$$\mu_A(1-k) = \mu_A(k-1) = k/2 \ge 1/2 = \mu_A(0).$$

Thus, we have that for $k > 3$

$$\mu_{f^{cz1}(A)}(k) = \mu_{f^{cz2}(A)}(k) = \mu_{f^{cz3}(A)}(k) = \mu_A(0) = 1/2,$$

for $2 \le k \le 3$

$$\mu_{f^{cz1}(A)}(k) = \mu_{f^{cz2}(A)}(k) = \mu_{f^{cz3}(A)}(k) = 1/(k-1),$$

for $1 < k \le 2$

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and that for $k < 1$

$$\mu_{f^{cz1}(A)}(k) = \mu_{f^{cz2}(A)}(k) = \mu_{f^{cz3}(A)}(k) = \mu_A(0) = 1/2.$$

Example 4. Let the function $f: \mathbb{R} \to \mathbb{R}$ (see figure 3) be defined by

$$f(x) = \begin{cases} x^2/|x| + 1 & \text{for } 0 < |x| \le 2, \\ k(\ne 1) & \text{for } x = 0, \\ -|x| + 5 & \text{for } |x| \ge 2. \end{cases}$$

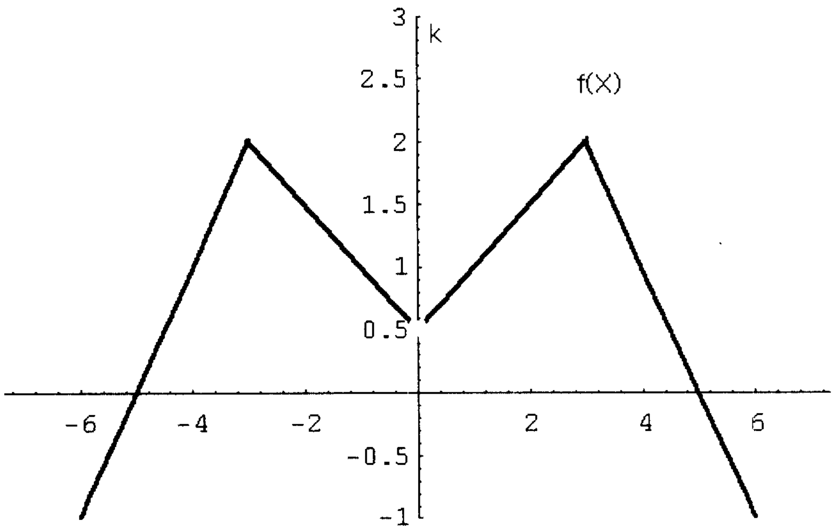

figure 3

Consider $A \in \tilde{P}(\mathbb{R})$ given in the example 3. Note that $f^{-1}(1) = \{-4, 4\}$ and $S^1 = \{-4, 0, 4\}$. Thus, we have

$$\mu_{f^{CZ1}(A)}(1) = \mu_{f^{CZ2}(A)}(1) = 1/4, \quad \mu_{f^{CZ3}(A)}(1) = 1/2.$$

Note that for $k \in \mathbb{R} - \{1\}$ $S^k \subset f^{-1}(k)$. More precisely, if $k > 3$ then

$$f^{-1}(k) = \{0\} \text{ and } S^k = \phi,$$

if $1 < k \leq 3$, then

$$f^{-1}(k) = \{k-5, k-1, 1-k, 5-k\} = S^k,$$

and if $k < 1$, then

$$f^{-1}(k) = \{k-5, 5-k\} = S^k.$$

By the proposition 3.4, we see that for $k \in \mathbb{R} - \{1\}$

$$\mu_{f^{CZ1}(A)}(k) = \mu_{f^{CZ2}(A)}(k) = \mu_{f^{CZ3}(A)}(k).$$

Definition 3.5.(Extension of 3.3) Let f be a function from $D \subset \mathbb{R}$ to $\mathbb{R}$ and let f have the limit a' at each $a \in \mathbb{R} - D$. If $A \in \tilde{P}(\mathbb{R})$, then the fuzzy sets $f(A)$ in $\mathbb{R}$ can be characterized by three ways:

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(Method 3).

$$\mu_{f(A)}(y) = \begin{cases} \sup_{x \in f^{-1}(y) \cup S^y} \mu_A(x) & \text{if } f^{-1}(y) \cup S^y \neq \phi, \\ 0 & \text{if } f^{-1}(y) \cup S^y = \phi. \end{cases}$$

The fuzzy sets $f(A)$ in $\mathbb{R}$ obtained by using methods 1, 2 and 3 will be called the *1st, 2nd 3rd enlarged C-Z(Choi-Zadeh) fuzzy extensions of* f, respectively and will be denoted by $\bar{f}^{CZ1}(A)$, $\bar{f}^{CZ2}(A)$ and $\bar{f}^{CZ3}(A)$, respectively.

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