A STUDY ON PROCESS CAPABILITY INDICES FOR NON-NORMAL DATA

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Abstract

Quality characteristics on the properties of process capability indices (PCIs) are often required to be normally distributed. But, if a characteristic is not normally distributed, serious errors can result from normal-based techniques. In this case, we may well consider the use of new PCIs specially designed to be robust for non-normality.

In this paper, a newly proposed measure of process capability is introduced and compared with existing PCIs using the simulated non-normal data.

1. Review of PCI

Several authors have promoted the use of various process capability indices and examined their statistical properties. In this section, some theoretically well-established PCIs will be reviewed with brief remarks on their properties.

They are all given as a form of ratio relating a measure of allowable process spread to a measure of actual process spread. The resultant ratio is unitless, thereby allowing comparisons to be drawn across a broad spectrum of processes.

We now proceed to define and examine the earliest form of PCI , generally denoted by C_p .

$$C_p = \frac{USL - LSL}{6\sigma}$$

where USL and LSL denote the upper and lower specification limits, respectively, and σ the process standard deviation. Clearly, large values of C_p are desirable and the value $C_p = 1$ indicates that it is possible to have the expected proportion of NC ("Nonconforming", values of characteristic X outside specification limits) product as small as 0.27%. Montgomery (1985) cites recommended minimum values for C_p , as followes :

- for an existing process, 1.33
- for a new process, 1.50

But, C_p fails to take into consideration proximity to the target value in its assessment of a process capability as is easily seen in the form of C_p . Due to the inherent inability of C_p to consider targets, several indices have been proposed that attempt to take the target value T into account. This class of indices includes

$$C_{pl} = \frac{\mu - LSL}{3\sigma}$$

$$C_{pu} = \frac{USL - \mu}{3\sigma}$$

$$C_{pk} = \min(C_{pl}, C_{pu})$$

$$C_{pk}^* = (1-k)\, C_p$$

with notations in Kane(1986) where $k = \frac{2|T-\mu|}{USL-LSL}$ and process mean μ satisfies the condition $LSL < \mu < USL$. C_{pk} and C_{pk}^* are often treated interchangeably; they are numerically equivalent when $0 \le k \le 1$, and the target value is the midpoint of the specification limits.

Also belonging to this class of indices is C_{pm} , which is defined as (in the published literature by Chan *et al.* (1988a))

$$C_{pm} = \frac{d}{3\sqrt{\sigma^2 + (\mu - T)^2}}$$

where $d = \frac{USL-LSL}{2}$ denotes the half length of the specification limits.

The motivation for C_{pk} , C_{pk}^* , and C_{pm} arises directly from the inability of C_p to consider the target value. However, C_{pk} and C_{pk}^* are substantially different in their determination from either C_p or C_{pm} , while C_{pm} represents a minor alteration of C_p , essentially examining squared deviations from the target rather than from the process mean. C_{pl} and C_{pu} are unilateral measures of process capability as they consider only a single specification limit.

C_{pk} is obtained from C_p by modifying the numerator and C_{pm} is obtained from C_p by modifying the denominator. If the two modifications are combined, we obtain the index

$$C_{pmk} = \frac{d - |\mu - m|}{3\{\sigma^2 + (\mu - T)^2\}^{\frac{1}{2}}}$$

introduced by Pearm *et al.*(1992).

All process capability indices of this form are determined under the assumptions that

(a) the process is in control

(b) the process measurements are normally distributed.

2. Proposal of a New Measure for Process Capability

All indices proposed in the previous section are studied and developed under the assumption of normality. Although some practitioners have been well aware of possible problems that can be caused by deviation from the normality of a quality characteristic, X, the effects of non-normality on properties of PCIs have not been a major research item until quite recently.

The discussion of non-normality so far falls into two parts. The first is investigation of the properties of PCIs and their estimators when the distribution of X has specific non-normal forms. The second is development of methods allowing non-normality and consideration of new PCIs specially designed to be robust to non-normality. Some robust PCIs to reduce the effects of non-normality are described and discussed in Kotz and Johnson (1993).

Usual approach is either to make a direct allowance for the values of the skewness and kurtosis coefficients or to get limits which are insensitive to these values. Another approach, aimed at enhancing connection between PCI values and expected proportions of NC, tries to correct the PCI, so that the corrected value corresponds (at least approximately) to what would be the value for a normal process distribution with the same expected proportion NC.

The newly proposed PCI, which will be denoted as C_{pn} (subscript n refers to non-normal), also accommodates this idea. And the two different lengths $\mu - LSL$

and $USL-\mu$ for non-normally distributed data especially for skewed distribution must be taken into account so that they can influence the magnitude of the index altogether.

The new measure C_{pn} to evaluate process capability is as follows. For a given non-normally distributed distribution f(x) with cumulative density funtion(cdf) F(x),

let q_1=1-F(LSL), q_2=F(USL)

$$Z_l = 6 - \Phi^{-1}(q_1)\ ,\quad Z_u = 6 - \Phi^{-1}(q_2)$$

where Φ is the inverse cdf of standardized normal distribution.

Then

$$C_{pn} = w_1 \frac{\mu - LSL}{Z_l \cdot \sigma} + w_2 \frac{USL - \mu}{Z_u \cdot \sigma}$$

where w_i's are weights with $w_1 + w_2 = 1$.

This measure based on the perception that C_p for normal case can be divided into two parts and weighted so that it is written as $\frac{1}{2} \cdot \frac{d}{3\sigma} + \frac{1}{2} \cdot \frac{d}{3\sigma}$, where 'd' in the numerator , '3' in the denominator and weight $\frac{1}{2}$ are to be properly modified for non-normally distributed data to meet the same expected NC proportion idea and to take the two different lengths of the specification into consideration. Therefore, C_{pn} becomes C_p exactly when the quality characteristic is under normal distribution and the process mean is centered at the target value.

Among the possible choices of w_1 and w_2, $w_1 = \frac{\mu - LSL}{USL - LSL}$, $w_2 = \frac{USL - \mu}{USL - LSL}$ are chosen because we hope that the longer is the length of the specification limit, the more it needs to be reflected in the index. Z_l and Z_u are designed to describe the fact that the larger is the expected NC proportion , the smaller the index needs to be.

In the following section, some cases are simulated and compared with efficiency ratios using the MSEs (mean square errors) of the indices from the generated non-normally distributed sample data.

3. Simulation and Comparison

Simulations were carried out to compare the behaviors of the proposed index with those of the other indices already reviewed in section 1 using gamma, beta and t distributions. A random variable X has the gamma distribution $\Gamma(\alpha, \beta)$ if its density is $\frac{1}{\Gamma(\alpha)\beta^{\alpha}} x^{\alpha-1} e^{-\frac{x}{\beta}}$, $x>0$, $\alpha>0$, $\beta>0$ with respect to Lebesque measure on $(0, \infty)$ which will be used in this section as a representative for one side bounded density function.

Beta and t distributions with densities $B_{\alpha,\beta}(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}$ for $0<x<1$, $t_3(x) = \frac{1}{\sqrt{3\pi}\Gamma(1.5)}(1+\frac{x^2}{3})^{-2}$ will be used as a representative for bounded and unbounded density function, respectively. Detailed account and figures are available in Bickel and Doksum (1976).

The criterion is the robustness of the index for non-normality. To verify the better performances of C_{pn} over the other indices, we define the ratios of MSEs as follows.

$$ER_1 = \frac{MSE(\widehat{C_p})}{MSE(\widehat{C_{psk}})}$$

$$ER_2 = \frac{MSE(\widehat{C_{pk}})}{MSE(\widehat{C_{psk}})}$$

$$ER_3 = \frac{MSE(\widehat{C_{pm}})}{MSE(\widehat{C_{psk}})}$$

$$ER_4 = \frac{MSE(\widehat{C_{pmk}})}{MSE(\widehat{C_{psk}})}$$

where ER_1, ER_2, ER_3 and ER_4 refer to the efficiency ratio of C_{pn} over C_p , C_{pk} , C_{pm} and C_{pmk}, respectively. For the calculation of the indices, only non-normally distributed sample date are used through the Shapiro and Wilk's W-test for normality with significant probability less than 0.1. Shapiro and Wilk's(1965) W statistics has been shown to provide a superior omnibus test of normality(Pearson *et al.*, 1977).

Tables 3.1-9 are the summarization of the simulation results. Each table contains the MSEs of empirically calculated PCIs and ERs. 500 times iteration was done for each sample size n=10, 20, 50, 100 and effective iteration numbers are also noted right beside each sample size obtained by excluding the normally distributed data through the Shapiro and Wilk's W-test for normality.

Tables 3.1-3 show the behaviours of C_{pn} over C_p, C_{pk}, C_{pm} and C_{pmk} in case of bilateral specification limits available. Even though there does not seem to exist any monotonic relation between sample size n and ER , C_{pn} behaves uniformly good over the other indices (*i.e* all ERs are larger than 1) regardless of sample size n. And there is extraordinary good performance of C_{pn} over C_{pmk} as is easily seen in the tables. So we may say that it is desirable to adopt C_{pn} as a robust PCI for non-normally distributed data in case of bilateral specification limits are available.

Meanwhile in case of unilateral specification limit available, the performances of C_{pn} depend on the distributions from which the sample data are generated and which side of specification limit is used. For the three distributions (USL only) in Tables 3.4-6, C_{pn} shows satisfactory result over the other indices except C_{pm} for n=10 (gamma).

And for LSL only cases in Tables 3.7-8, C_{pn} seem to be acceptable for t and gamma distribution even though we have ER_3=0.95 for n=10 (gamma). But, for beta distribution in Table 3.9(LSL only), C_{pn} behaves quite differently. Unlike its better performance for the USL only case, C_{pn} does not seem to show much stabler behavior than the other indices for the LSL only case. It is much worse than C_{pmk} even for large sample size n=100. So the use of C_{pn} for the LSL only case is not recommendable for beta distribution.

In addition, when the quality characteristic is under t distribution, that is, kurtosis may exist, ERs are always around 1.3 in Tables 3.1, 3.4, 3.7. In other words, C_{pn} behaves overwhelmingly well when the distribution may have high probability in tail areas.

Table 3.1 Simulation results of MSE (ER) for t(3)
($\pm 2\sigma$ specification limits)

	n			
	10(132)	20(220)	50(300)	100(381)
C_p	0.354 (1.34)	0.231 (1.34)	0.160 (1.32)	0.131 (1.32)
C_{pk}	0.361 (1.37)	0.232 (1.34)	0.160 (1.32)	0.130 (1.31)
C_{pm}	0.344 (1.30)	0.225 (1.30)	0.158 (1.31)	0.130 (1.31)
C_{pmk}	0.354 (1.34)	0.238 (1.38)	0.166 (1.37)	0.131 (1.32)
C_{pn}	0.264	0.173	0.121	0.099

Table 3.2 Simulation results of MSE (ER) for $\Gamma(9, 0.1)$
($\pm 2\sigma$ specification limits)

	n (effective iteration number)			
	10(65)	20(109)	50(238)	100(361)
C_p	0.267 (1.27)	0.114 (1.30)	0.072 (1.31)	0.050 (1.28)
C_{pk}	0.179 (1.10)	0.116 (1.32)	0.075 (1.36)	0.051 (1.31)
C_{pm}	0.163 (1.00)	0.108 (1.23)	0.071 (1.29)	0.049 (1.26)
C_{pmk}	0.228 (1.40)	0.152 (1.73)	0.101 (1.84)	0.068 (1.74)
C_{pn}	0.163	0.088	0.055	0.039

Table 3.3 Simulation results of MSE (ER) for $B(7,4)$
($\pm 2\sigma$ specification limits)

	n (effective iteration number)			
	10(53)	20(73)	50(124)	100(222)
C_p	0.179 (1.20)	0.099 (1.24)	0.065 (1.27)	0.045 (1.32)
C_{pk}	0.159 (1.07)	0.100 (1.27)	0.064 (1.25)	0.047 (1.38)
C_{pm}	0.150 (1.01)	0.091 (1.15)	0.062 (1.22)	0.044 (1.29)
C_{pmk}	0.177 (1.19)	0.117 (1.46)	0.075 (1.47)	0.048 (1.41)
C_{pn}	0.148	0.079	0.051	0.034

Table 3.4 Simulation results of MSE (ER) for t(3)
($w_1 = 0$, $w_2 = 1$, $\pm 3\sigma$ specification limits)

	n (effective iteration number)			
	10(220)	20(300)	50(132)	100(381)
C_p	0.365 (1.32)	0.246 (1.32)	0.168 (1.31)	0.131 (1.31)
C_{pk}	0.373 (1.35)	0.247 (1.33)	0.168 (1.31)	0.131 (1.31)
C_{pm}	0.354 (1.28)	0.238 (1.28)	0.166 (1.30)	0.131 (1.31)
C_{pmk}	0.387 (1.40)	0.270 (1.45)	0.185 (1.45)	0.138 (1.38)
C_{pn}	0.277	0.186	0.128	0.100

Table 3.5 Simulation results of MSE (ER) for $\Gamma(9, 0.1)$

($w_1 = 0$, $w_2 = 1$, $\pm 3\sigma$ specification limits)

	n (effective iteration number)			
	10(65)	20(109)	50(238)	100(361)
C_p	0.390 (1.18)	0.215 (1.18)	0.138 (1.18)	0.094 (1.19)
C_{pk}	0.357 (1.08)	0.213 (1.17)	0.137 (1.17)	0.092 (1.18)
C_{pm}	0.305 (0.92)	0.202 (1.11)	0.134 (1.15)	0.091 (1.15)
C_{pmk}	0.382 (1.16)	0.257 (1.41)	0.170 (1.45)	0.115 (1.46)
C_{pn}	0.330	0.182	0.117	0.079

Table 3.6 Simulation results of MSE (ER) for B(3,7)

($w_1 = 0$, $w_2 = 1$, $\pm 3\sigma$ specification limits)

	n (effective iteration number)			
	10(82)	20(103)	50(233)	100(395)
C_p	0.323 (1.09)	0.177 (1.09)	0.118 (1.08)	0.085 (1.09)
C_{pk}	0.316 (1.07)	0.165 (1.01)	0.111 (1.02)	0.082 (1.05)
C_{pm}	0.299 (1.01)	0.165 (1.01)	0.112 (1.03)	0.082 (1.05)
C_{pmk}	0.352 (1.19)	0.200 (1.23)	0.143 (1.31)	0.107 (1.37)
C_{pn}	0.296	0.163	0.109	0.078

Table 3.7 Simulation results of MSE (ER) for t(3)
($w_1 = 1$, $w_2 = 0$, $\pm 3\sigma$ specification limits)

	n (effective iteration number)			
	10(220)	20(300)	50(132)	100(381)
C_p	0.371 (1.32)	0.240 (1.32)	0.165 (1.32)	0.137 (1.32)
C_{pk}	0.378 (1.35)	0.242 (1.33)	0.165 (1.32)	0.137 (1.32)
C_{pm}	0.358 (1.27)	0.234 (1.29)	0.163 (1.30)	0.136 (1.31)
C_{pmk}	0.346 (1.23)	0.225 (1.24)	0.158 (1.26)	0.130 (1.25)
C_{pn}	0.281	0.182	0.125	0.104

Table 3.8 Simulation results of MSE (ER) for $\Gamma(9, 0.1)$
($w_1 = 1$, $w_2 = 0$, $\pm 3\sigma$ specification limits)

	n (effective iteration number)			
	10(65)	20(109)	50(238)	100(361)
C_p	0.171 (1.11)	0.110 (1.18)	0.064 (1.10)	0.049 (1.11)
C_{pk}	0.154 (1.00)	0.115 (1.17)	0.071 (1.22)	0.053 (1.20)
C_{pm}	0.146 (0.95)	0.106 (1.11)	0.064 (1.10)	0.049 (1.11)
C_{pmk}	0.163 (1.06)	0.108 (1.41)	0.071 (1.22)	0.049 (1.11)
C_{pn}	0.154	0.100	0.058	0.044

Table 3.9 Simulation results of MSE (ER) for B(7,3)

($w_1 = 1$, $w_2 = 0$, $\pm 3\sigma$ specification limits)

	n (effective iteration number)			
	10(57)	20(111)	50(238)	100(405)
C_p	0.679 (1.09)	0.190 (1.09)	0.122 (1.09)	0.084 (1.09)
C_{pk}	0.588 (0.92)	0.168 (0.96)	0.110 (0.98)	0.082 (1.07)
C_{pm}	0.290 (0.45)	0.164 (0.94)	0.113 (1.01)	0.082 (1.07)
C_{pmk}	0.241 (0.38)	0.141 (0.81)	0.093 (0.82)	0.065 (0.84)
C_{pn}	0.640	0.175	0.112	0.077

4. Concluding Remarks

Most SPC techniques require that the variable quality characteristics be at least approximately normally distributed. Some well-known PCIs, however, are sensitive to departure from normality so that the use of those indices are not recommended. In this paper, C_{pn} ,a newly proposed measure of process capability, is introduced by taking both sides $\mu - LSL$ and $USL - \mu$ into consideration and with some properly chosen Z_l , Z_u and weights.

Through the simulation study for Gamma, Beta and t distribution , C_{pn} gives us a quite good performance when bilateral specification limits are available regardless of the sample size n from 10 to 100. Meanwhile in case of unilateral specification limit is available, the performances of C_{pn} depend on the distributions from which the sample data are generated and which side of specification limit is used. When quality characteristic is under beta distribution and only lower specification limit is available, the use of C_{pn} does not seem to be appropriate. In other cases, however, the performances of C_{pn} are mostly satisfactory.

To improve the performances of C_{pn} , some further research works are necessary in choosing proper Z_l , Z_u and weights.

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