퍼지 L-수렴 공간
Fuzzy L-Convergence Spaces

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ABSTRACT: We introduce a notion of fuzzy L-convergence of filters and show that the collection of fuzzy L-convergence spaces forms a cartesian closed topological category.

Many researchers studied on the convergence of prefilters or fuzzy filters on a set.(see [2]) In this paper we introduce a notion of 'fuzzy' convergence of filters on a set as one of attempts to get rid of gaps between mathematics and real world problem. For categorical notions, we refer to [1].

Let L be a complete Heyting algebra with a top element 1 and a bottom element 0. For a set X, $F(X)$ is the collection of all filters on X.

Definition 1 Let X be a set. A map $l : F(X) \times X \to L$ is called a *fuzzy L-convergence structure* on X if it satisfies the following conditions:

(1) $l(\dot{x}, x) = 1$ for all $x \in X$

(2) if $\mathcal{F} \subseteq \mathcal{G}$, then $l(\mathcal{F}, x) \leq l(\mathcal{G}, x)$ for all $x \in X$.

The pair (X, l) is called a *fuzzy L-convergence space.*

Definition 2 A map $f : (X, l) \to (Y, m)$ is said to be *continuous* if $l(\mathcal{F}, x) \leq m(f(\mathcal{F}), f(x))$

We form a category FLC consisting of all fuzzy L-convergence spaces and all continuous maps between them. The pair (X, l) is called a *fuzzy L-convergence space.*

Theorem 3 *The category FLC is topological.*

Proof. Let X be a set and $(X_\alpha, l_\alpha) \in \underline{FLC}$ for each $\alpha \in \Lambda$. Let $f_\alpha : A \to (X_\alpha, l_\alpha)$ be a map for each $\alpha \in \Lambda$. Define

$$l : F(X) \times X \to L \ \text{ by } \ l(\mathcal{F}, x) = \inf_\lambda l_\alpha(f_\alpha(\mathcal{F}), f(x)).$$

Then $l(\dot{x}, x) = 1$. Let $\mathcal{F} \subseteq \mathcal{G}$ in $f(X)$. Then

$$l(\mathcal{F}, x) = \inf_\lambda l_\alpha(\mathcal{F}, f_\alpha(x)) \leq \inf_\lambda l_\alpha(f_\alpha(\mathcal{G}), f_\alpha(x)) = l(\mathcal{G}, x).$$

Hence $(X, l) \in \underline{FLC}$.

Let $(Z, m) \in \underline{FLC}$ and $g : Z \to X$ be a map. Suppose $f_\alpha \circ g$ is continuous for all $x \in \Lambda$. Then $m(\mathcal{A}, z) \leq l_\alpha(f_\alpha(g(\mathcal{A})), f_\alpha(g(z)))$ for each $(\mathcal{A}, z) \in F(Z) \times Z$ and $\alpha \in \Lambda$. Hence $m(\mathcal{A}, z) \leq l(g(\mathcal{A}), g(z))$.

We note that the category $\underline{FLC}$ satisfies the fibre small condition and it has the terminal separator property. Λ

Remark 1 Let $(X_1, l_1), (X_2, l_2) \in \underline{FLC}$. Let $l : F(X_1 \times X_2) \times X_1 \times X_2 \to L$ be a map defined by $l(\mathcal{H}, (x_1, x_2)) = l_1(\pi_1(\mathcal{H}), x_1) \wedge l_2(\pi_2(\mathcal{H}), x_2)$. Then l is the fuzzy L-convergence structure for the product space $X_1 \times X_2$ in $\underline{FLC}$.

Theorem 4 *The category $\underline{FLC}$ is cartesian closed.*

Proof. For (X, l) and (Y, m) in $\underline{FLC}$, let $C(X, Y)$ be the set of all continuous maps between them. For each $\mathcal{L} \in F(C(X, Y))$ and $f \in C(X, Y)$, let

$$u(\mathcal{L}, f) = \sup\{\alpha \in L | l(\mathcal{A}, x) \wedge \alpha \leq m(\mathcal{L}(\mathcal{A}), f(x)) \text{ for all } (\mathcal{A}, x) \in F(X) \times X\}$$

Note that $l(\mathcal{A}, x) \wedge u(\mathcal{L}, f) \leq m(\mathcal{L}(\mathcal{A}, x))$ for all $(A, x) \in F(X) \times X$. Clearly, $u(\dot{f}, f) = 1$. Let $\mathcal{L}_1 \subseteq L_2 \in F(C(X, Y))$, then from the formula $l(\mathcal{A}, x) \wedge \alpha \leq m(\mathcal{L}_1(\mathcal{A}), f(x)) \leq m(\mathcal{L}_2(\mathcal{A}), f(x))$, it is clear to see that $u(\mathcal{L}_1, f) \leq u(\mathcal{L}_2, f)$ for all $f \in C(X, Y)$. Therefore $(C(X, Y), u) \in \underline{FLC}$.

Let $\mathcal{H} \in F(X \times C(X, Y))$ and k be the fuzzy $L-$convergence structure for the product space $X \times C(X, Y)$. Then

$$\begin{aligned} k(\mathcal{H}, (x, f)) &= l(\pi_1(\mathcal{H}), x) \wedge u(\pi_2(\mathcal{H}), f) \\ &\leq m(\pi_2(\mathcal{H})(\pi_1(\mathcal{A})), f(x)) \\ &\leq m(ev(\mathcal{H}), f(x)). \end{aligned}$$

Hence $ev : X \times C(X,Y) \to Y$ is continuous. Let $h : (X,l) \times (Z,m) \to (Y,m)$ be a map in FLC. Define $h^* : (Z,m) \to (C(X,Y),u)$ by $h^*(z)(x) = h(x,z)$. Since FLC has the terminal separator property, $h^*(z) \in C(X,Y)$ for all $z \in Z$. Take any $(\mathcal{G},z) \in F(Z) \times Z$ and $(\mathcal{A},x) \in F(X) \times X$. Since h is continuous, $l(\mathcal{A},x) \wedge n(\mathcal{G},z) \leq m(h(\mathcal{A} \times \mathcal{G}), h(x,z)) = m(h^*(\mathcal{G})(\mathcal{A}), h^*(z)(x))$. Hence $n(\mathcal{G},z) \leq u(h^*(\mathcal{G}), h^*(z))$. Therefore h^* is continuous. The uniqueness of such a map h^* is obvious. Λ

Remark 2 1.To the definition of a fuzzy L-convergence space, we may add the condition

$$(3) \text{ for } \mathcal{F},\mathcal{G} \in F(X), l(\mathcal{F},x) \wedge l(\mathcal{G},x) \leq l(\mathcal{F} \cap \mathcal{G},x)$$

without any change in the above results.

2. As a next step, we wil try to show that FLC is a quasitopos. Since our notion of 'fuzzy' convergence seems to be very useful in many applications, there are many other aspects to investigate.

References

[1] H.Herrlich, Cartesian closed topological categories, Math.Collog.Univ. Cape Town 9 (1974), 1-16

[2] K.C. Min, Fuzzy limit spaces, Fuzzy Sets and Systems 32 (1989), 343-357