Properties of the T-Fuzzy factor Groups

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1. Introduction .

The concept of fuzzy subgroup appeared in the pioneering paper [5] of Rosenfeld and J. M. Authony and H. Sherwood studied T-fuzzy group and it product. since then, papers on T-fuzzy group appear so many. In this paper we introduce the concept of T-fuzzy factor group and study its property.

2. Preliminaries

Definition 2.1 A t-norm is a function $T: [0,1] \times [0,1] \to [0,1]$ satisfy the following properties :

1) $T(x, 1) = x.$
2) $T(x, y) > T(x, z)$ if $y > z.$
3) $T(x, y) = T(z, u)$, if $x \geq z$ and $y \geq u$
4) $T(x, T(y, z)) = T(T(x, y), z)$

for all x, y, z and u in $[0,1]$

Definition 2.2 A function $A: G \to [0, 1]$ is a T-fuzzy subgroup of G if there is a t-norm T such that for all $x, y \in G$:

1) $A(xy) > T(A(x), A(y))$
2) $A(x^{-1}) = A(x)$
3) $A(e) = 1$ where e is the identity of G

where G denote a group whore operation is suppressed and indicated by juxtaposition.

Definition 2.3 For $i = 1$ and 2. Let A_i be a T-fuzzy subgroup of G_i. If there is a group homomorphism (isomorphism) f from G_1 onto G_2 such that $f(A_1) = A_2$ then we call A_1 and A_2 are homomorphic (isomorphic).

Definition 2.4 Let A be T-fuzzy subgroup of G. If $A(xy) = A(yx)$ holds for all x, y in G, then we call A the T-fuzzy normal subgroup of G.

Definition 2.5 Let A, B be separately fuzzy subsets of the nonempty set G. T a t-norm. The inner product of AB is the fuzzy subset of G defined by

$$AB(x) = \sup_{x = ab} T(A(a), B(b)), \ x \in G.$$

Definition 2.6 Let A, B be fuzzy sets of G, G' respectively T a t-norm. Then the direct product $A \times B$ is the fuzzy subset of $G \times G'$ defined by

$$A \times B(x, x') = T(A(x), B(x')), \ (x, x') \in G \times G'.$$

Definition 2.7 Let A be a T-fuzzy subgroup of G and $a \in G$. Then $aA(Aa)$ is called a left(right) fuzzy coset of A in G defined as following:

$(aA)(x) = A(a^{-1}x)$, for any $x \in G$.

It is clear A is normal iff $aA = Aa$ for all a in G.

Proposition 2.8 $aB = Ba$ iff $B(a^{-1}b) = B(e)$.

proposition 2.9 Let f be a homomorphism from G onto G'. T a continuous function. A a T-fuzzy subgroup of G, then $f(A)$ is a T-fuzzy subgroup of G'.

3. T-fuzzy factor group

Proposition 3.1 Let B be a T-fuzzy subgroup, a, b in G. Let $X = \{x : xB = aB\}$, $Y = \{x : xB = bB\}$, $Z = \{x : xB = abB\}$, then $Z = XY$.

Proposition 3.2 If B is normal, $aB = bB$, then $a^{-1}B = b^{-1}B$.

Definition 3.3 Let T be a t-normal. If T is a continuous function, then T is called a continuous t-norm.

In the Following we always assume T is a continuous t-norm. Let A be a T-fuzzy subgroup of G, B a T-fuzzy normal subgroup.

We define a fuzzy set on G/B as following

$$A/B: G/B \to [0,1], \quad A/B(aB) = \sup_{xB = aB} A(x), \text{ for any } ab \in G/B.$$

Proposition 3.4 The above A/B is a T-fuzzy subgroup of G/B.

Proof 1) $A/B((aB)(bB)) = A/B(abB)$

$$= \sup A(xy) \geq \sup T(A(x), A(y))$$
$$= T(\sup A(x), \sup A(y))$$
$$= T(A/B(aB), A/B(bB))$$

2) $A/B(a^{-1}B) = \sup A(x^{-1})$

$$= \sup A(x) = A/B(aB)$$

3) $A/B(B) = A(e) = 1$

Definition 3.5 We call the above A/B is T-fuzzy factor group of A with respect to B.

Proposition 3.6 Let A and A' be T-fuzzy subgroup of G, G'. Respectively, η a homomorphism from G onto G' satisfying $\eta(A) = A'$. Let B be a T-fuzzy normal subgroup of G satisfying $\ker \eta = G_B = \{x: B(x) = 1\}$. Then we can get an isomorphism f from G/B onto G' satisfying

$$f(A/B) = A' = \eta(A)$$

Proof Let $f: G/B \to G'$, $f(xB) = \eta(x)$.

If $aB = bB$, then $B(b^{-1}a) = B(ab^{-1}) = 1$

Hence $ab^{-1} \in G_B = \ker \eta$

$$\begin{aligned} \eta(a) &= \eta(ab^{-1}b) \\ &= \eta(ab^{-1})\eta(b) \\ &= \eta(b) \end{aligned}$$

Hence f is an one valued mapping

It is clear f is a homomorphism

If $\eta(x) = \eta(y)$, then $\eta(x) = \eta(xx^{-1}y) = \eta(x)\eta(x^{-1}y)$

so $\eta(x^{-1}y) = e$, $x^{-1}y \in \ker \eta = G_B$
$B(x^{-1}y) = 1$, $xB = yB$

Hence f is an isomorphism from G/B onto G'.

For any $x' \in G'$

$$f(A/B)(x') = \sup_{\eta(y) = x'} A/B(yB) = \sup_{\eta(y) = x',\, zB = yB} A(z)$$

$$=\sup_{\eta(z)=x'} A(z) = \eta(A)(x') = A'(x')$$

Hence $f(A/B) = A'$

Proposition 3.7 Let A, B be T-fuzzy groups of G. C a T-fuzzy normal of G. Then $A \cap B/C = A/C \cap B/C$.

References

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