퍼지 미분방정식의 최적 제어 문제
OPTIMAL CONTROL PROBLEM FOR FUZZY DIFFERENTIAL EQUATIONS

YOUNG-CHEL KWUN[1], JAE-RONG CHOI[1], HOE-YOUNG HA[2], AND BU-YOUNG LEE[1]

1. Introduction

In this paper, we study the optimal control problem for the fuzzy differential system using by method of the Huhn-Tucker theorem ([1], [3]).

2. Fuzzy optimal control

In this section, we set up a problem of the form, in which the domain of the functional itself consists of uncertain objects.

For $u, v \in \mathcal{E}^n$, put

$$\rho_2(u,v)^2 = \int_0^1 \int_{S^{n-1}} |S_u(\beta,x) - S_v(\beta,x)|^2 d\mu(x) d\beta,$$

where $\mu(\cdot)$ is unit Lebesgue measure on S^{n-1}. In particular, define $\|u\| = \rho_2(u, \{0\})$. Observe that if n=1 and $[u]^\beta = [u_l(\beta), u_r(\beta)]$, then

$$\|u\|^2 = \int_0^1 (u_l(\beta)^2 + u_r(\beta)^2) d\beta.$$

Let $a(t) = [a_l(t), a_r(t)]$ and $f(t) = [f_l(t), f_r(t)]$ be a nonempty compact interval-valued function on $0 \le t \le T < \infty$ with a_l, a_r continuous on $[0,T]$. Denote by $\mathcal{E}^1_{Lip}([0,T])$ $\big(\text{resp.}\mathcal{E}^1([0,T])\big)$ the class of continuous $\mathcal{E}^1_{Lip}$-valued (resp.$\mathcal{E}^1$-valued) functions on $[0,T]$.

Consider the following fuzzy control system

$$\text{(FCS)} \qquad \begin{cases} \dot{x}(t) = a(t)x(t) + f(t) + u(t), \\ x(0) = x_0, \end{cases}$$

1. Dept. of Math. Dong-A Univ.
2. Dept. of Math. In-Je Univ.

where state $x(\cdot)$ and control $u(\cdot)$ in $\mathcal{E}^1([0,T])$.

Our problem is to minimize

(1) $$J(\omega) = \frac{1}{2}\int_0^T \|u(t)\|^2 dt$$

subject to

(2) $$x(T) \succeq_\alpha x^1, x^1 \in \mathcal{E}^1.$$

The function $t \longmapsto a(t)x(t)$ is Lipschitz ,

$$\rho_2(ax, ay) \le \max_{t\in[0,T]}\{|a_l(t)|, |a_r(t)|\}\rho_2(x,y).$$

Then (FCS) has a unique solution on [0,T] for a given continuous control $u(\cdot)$([3]). For given u, the trajectory $x(t)$ is represented by

$$x(t) = S(t)x_0 + \int_0^t S(t-s)f(s)ds + \int_0^t S(t-s)u(s)ds, \quad 0 \le t \le T$$

where $S(t)$ is an interval-valued function

$$[S_l(t), S_r(t)] = [e^{\int_0^t a_l(s)ds}, e^{\int_0^t a_r(s)ds}].$$

Write $[x(t)]^\beta = [x_l(\beta,t), x_r(\beta,t)]$ with a like notation for $u(t)$.

Let P be the positive orthant in R^n. For a given $\alpha \in I$, defined $\mathcal{P}_\alpha \subset \mathcal{E}^n_{Lip}$ by

$$\mathcal{P}_\alpha = \{u \in \mathcal{E}^n_{Lip} : [u]^\alpha \subset P\}.$$

If $u \in \mathcal{P}_\alpha$, write $u \succeq_\alpha 0$ and if $u -_h v \succeq_\alpha 0$ write $u \succeq_\alpha v$, where $-_h$ is Hukuhara difference and $u \succeq_\alpha 0$ if and only if $u \ge 0$ with necessity α. The positive dual cone of $\mathcal{P}_\alpha$ is the closed convex cone $\mathcal{P}_\alpha^\oplus \subset \mathcal{E}^{n^*}_{Lip}$, defined by

$$\mathcal{P}_\alpha^\oplus = \{p \in \mathcal{E}^{n^*}_{Lip} : < u, p > \ge 0 \text{ for all } u \in \mathcal{P}_\alpha\},$$

where $< u, p >= p(u)$ is the value at $u \in \mathcal{E}^n_{Lip}$ of the linear functional $p : \mathcal{E}^n_{Lip} \to R$, the space of which is denoted by $\mathcal{E}^{n^*}_{Lip}$.

The significance of $\mathcal{P}_\alpha^\oplus$ in our problem is that Lagrange multipliers for local optimization live in this dual cone. Local necessity coditions for (1) and (2) are more accessible when there is some notation of differentiability of the uncertain constraint functions $G = x(t) -_h x^1$. Let $\Pi : \mathcal{E}^1_{Lip} \to C(I \times S^0)$ be the canonical embedding. where $S^0 = \{-1, +1\}$. The fuzzy function G is said to be (Fréchet) differentiable at ξ_0 if the map $\widehat{G} = \Pi \circ G$ is Fréchet differentiable at ξ_0. A point ξ_0 is said to be a regular point of the uncertain constraint $G(\xi) \succeq_\alpha 0$ if $G(\xi_0) \succeq_\alpha 0$ and there is $h \in R$ such that $\widehat{G}(\xi_0) + D\widehat{G}(\xi_0)h \succeq_\alpha 0$. Our constraint function be compact-interval valued, $G(\xi) = [G_l(\xi), G_r(\xi)]$ and $J : R^n \to R$. The support function of $G(\xi)$ is

$$\Pi\big(G(\xi)\big)(x) = S_{G(\xi)}(x) = \begin{cases} -G_l(\xi) & \text{if } x = -1, \\ +G_r(\xi) & \text{if } x = +1, \end{cases}$$

since $S^0 = \{-1, +1\}$. Then $\Pi \circ g = S_{G(\cdot)}$ is obviously differentiable if and only if G_l, G_r are differentiable, and $S'_{G(\xi)}(-1) = -\nabla G_l(\xi)$, $S'_{G(\xi)}(+1) = \nabla G_r(\xi)$. The element of $\mathcal{P}_\alpha^\oplus$ can be seen to be of the form $l_0\lambda_0 + l_{+1}\lambda_{+1} + l_{-1}\lambda_{-1}$, where l_i are nonnegative constants, the λ_i map S^0 to R and $\lambda_{+1}(-1) = \lambda_{-1}(+1) = 0$, $\lambda_{+1}(+1) \geq 0$, $\lambda_{-1}(-1) \leq 0$ and $\lambda_0(-1) = \lambda_0(+1) \geq 0$. So each element of $\mathcal{P}_\alpha^\oplus$ acts like a triple of nonnegative constants $(\lambda_{-1}, \lambda_0, \lambda_{+1})$,

$$\lambda^*\big(S_{G(\xi)}(\cdot)\big) = (\lambda_{-1} - \lambda_0)G_l(\xi) + (\lambda_0 + \lambda_{+1})G_r(\xi),$$

which is always nonneagtive since $\lambda_0\big(G_r(\xi) - G_l(\xi)\big) \geq 0$. If ξ_0 is a regular point which is a solution to the constrained minimization, the Kuhn-Tucker conditions, namely that there exists $\lambda^* \geq 0$ so that

$$\begin{aligned} \nabla J(\xi_0) + \lambda^*\big(S'_{G(\xi_0)}(\cdot)\big) &= 0 \\ \lambda^*\big(S_{G(\xi_0)}(\cdot)\big) &= 0 \end{aligned}$$

can be written as

$$\begin{aligned} \nabla J(\xi_0) + (\lambda_{-1} - \lambda_0)\nabla G_l(\xi_0) + (\lambda_0 + \lambda_{+1})\nabla G_r(\xi_0) &= 0 \\ (\lambda_{-1} - \lambda_0)G_l(\xi_0) + (\lambda_0 + \lambda_{-1})G_r(\xi_0) &= 0. \end{aligned}$$

for some nonnegative reals $\lambda_{-1}, \lambda_0, \lambda_{+1}$. This extends quite naturally to a fuzzy real number constraint with necessity α, as follows :

Define the function $G : R^n \to \mathcal{E}^1$ by

$$[G(\xi)]^\alpha = [G_l(\alpha, \xi), G_r(\alpha, \xi)],$$

where for each $\xi \in R^n$, $G_l(\cdot, \xi)$ is monotone, nondecreasing in α and $G_r(\cdot, \xi)$ is monotone, nondecresasin in α (since $\alpha \leq \beta$ implies that $[G(\xi)]^\beta \subseteq [G(\xi)]^\alpha$. Suppose further that $G_l(\alpha, \cdot)$ and $G_r(\alpha, \cdot)$ are differentiable in ξ for each $\alpha \in I$. Write, for each fixed α,

$$G(\xi) \succeq_\alpha 0 \text{ if and only if } [G(\xi)]^\alpha \succeq 0.$$

Then, if ξ_0 is a regular point of the constraint $G(\xi) \succeq_\alpha 0$ minimizing $J(\omega)$, there exist nonnegative real numbers $\lambda_{-1}, \lambda_0, \lambda_{+1}$ satisfying

$$\begin{aligned} \nabla J(\omega) + (\lambda_{-1} - \lambda_0)\nabla_\xi G_l(\alpha, \xi_0) + (\lambda_0 + \lambda_{+1})\nabla_\xi G_r(\alpha, \xi_0) &= 0 \\ (\lambda_{-1} - \lambda_0)G_l(\alpha, \xi_0) + (\lambda_0 + \lambda_{+1})G_r(\alpha, \xi_0) &= 0. \end{aligned}$$

Finally, we assume following statement

(F) For any given $T > 0$ and $\alpha \in [0,1]$, there exists some interval valued $M(T) = [M_l(T), M_r(T)]$ such that

$$M(T) = \int_0^T S(T-s)f(s)ds,$$
$$[M(T)]^\beta = [M_l(T,\beta), M_r(T,\beta)].$$

Theorem . *Under hypothesis (F), there exists fuzzy control $u_0(t)$ for the Fuzzy optimal control problem (1) and (2) such that*

$$\begin{aligned} J(u_0) &= min J(\omega) \\ &= \frac{1}{2T^2}\int_0^T \int_0^1 \Big[S_l(T-s)^{-2}\Big(x_l^1(\beta) - S_l(T)x_0^-(\beta) - M_l(\beta)\Big)^2 \\ &\qquad + S_r(T-s)^{-2}\Big(x_r^1(\beta) - S_r(T)x_0^+(\beta) - M_r(\beta)\Big)^2\Big] d\beta dt \end{aligned}$$

which is attained when

$$L_{-1}(\beta) = \frac{x_l^1(\beta) - S_l(T)x_0^-(\beta) - M_l(\beta)}{TS_l(T-s)^2},$$

$$L_{+1}(\beta) = \frac{x_r^1(\beta) - S_r(T)x_0^+(\beta) - M_r(\beta)}{TS_r(T-s)^2}.$$

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