ON FUZZY METRIC SPACE

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Abstract. In this papaers, we generalize the usual fuzzy metric on R, the set of all real numbers, and induce the fuzzy metric space(X, $\tilde{d}$) from a metric space(X,d).

1. Introduction.

In1, we had a fuzzy real line, the system $\mathbb{R}$ of all real numbers together with its usual fuzzy metric and its usual fuzzy topology. In this paper, we obtain the definition of induced metric (Definition 3.1.) using similar methods in[1], induced a fuzzy metric topology (Definition 3.7.) from induced fuzzy metric space and have some properties of them.

2. Preliminaries.

Thoughtout this paper, the closed unit interval[0,1] in the real line $\mathbb{R}$ will be denoted by I, which I_0=(0,1) and $\mathbb{R}^+ = [0,\infty)$.

And the motation will be used as in[4]. In paticular, F_p(X) denote the set of all fuzzy point in X, and $\widetilde{P}$(X) denote the fuzzy power set of X.

The support of A $\in \widetilde{P}$(X), denoted by $\mathcal{S}$(A), is the ordinary subset of X;

$$\mathcal{S}(A) = \{x \in X \,|\, \mu_A(x) > 0\} \qquad (2.1)$$

([2]). Thus, we may regard $\mathcal{S}$ as a mapping from $\widetilde{P}$(X) to P(X), the power set of X defined by (2.1)

Proposition 2.1. *The mapping $\mathcal{S} : \widetilde{P}(X) \longrightarrow P(X)$ satisfies the followings;*

(i) $\mathcal{S}(\bigcup_{j\in J} A_j) = \bigcup_{j\in J} \mathcal{S}(A_j)$ *for any* $\{A_j | j \in J\} \subset \widetilde{P}(X)$,

(ii) $\mathcal{S}(\bigcap_{j\in J} A_j) = \bigcap_{j\in J} \mathcal{S}(A_j)$ *for any* $\{A_j | j \in J\} \subset \widetilde{P}(X)$,

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX *(iii)* $\mathcal{S}(A^c)=\mathcal{S}A(1)^c$ *for any* $A \in \widetilde{P}(X)$, *where* $A(1) = \{x \in X | \mu_A(x) = 1\}$

Proposition 2.2. *([4]) Let A be a nonempty fuzzy set in X. Then;*

$$A = \bigcup\{x_\alpha \in F_p(x) | x \in \mathcal{S}(A), 0 < \alpha \le \mu_A(x)\}$$
$$= \bigcup\{x_\alpha \in F_p(x) | x \in \mathcal{S}(A), 0 < \alpha < \mu_A(x)\}.$$

Definition 2.3. *Define a relation* $\le$ *in* $F_p(\mathbb{R})$ *by for any* x_α, $y_\beta \in F_p(\mathbb{R})$ $x_\alpha \le y_\beta$ *if* $x < y$ *or* $x = y$ *and* $\alpha \le \beta$.

In particular, we write $x_\alpha < y_\beta$ if $x_\alpha \le y_\beta$ but $x_\alpha \ne y_\beta$. More presisely we write

$$x_\alpha =< y_\beta \quad if \quad x = y \quad and \quad \alpha < \beta, \tag{2.2}$$

$$x_\alpha <= y_\beta \quad if \quad x < y \quad and \quad \alpha = \beta, \tag{2.3}$$

$$x_\alpha << y_\beta \quad if \quad x < y \quad and \quad \alpha < \beta. \tag{2.4}$$

Of course, $x_\alpha \le< y_\beta$ means (2.2) or (2.4) and $x_\alpha <\le y_\beta$ means (2.3) or (2.4). We call the relation $\le$ in $F_p(\mathbb{R})$ the **usual fuzzy order** in $\mathbb{R}$.

Porposition 2.4. *The usual fuzzy order* $\le$ *in* $\mathbb{R}$ *is a linear order in* $F_p(\mathbb{R})$, *that is the following hold for any* x_α, y_β, $z_\gamma \in F_p(\mathbb{R})$.

(1) $x_\alpha \le x_\alpha$. *(reflexivity)*

(2) $x_\alpha \le y_\beta$ and $y_\beta \le x_\alpha$ *imply* $x_\alpha = y_\beta$ *(anti-symmetry)*

(3) $x_\alpha \le y_\beta$ and $y_\beta \le z_\gamma$ *imply* $x_\alpha = z_\gamma$ *(transitivity)*

(4) *Exactly one of the following holds;*

$x_\alpha < y_\beta$, $x_\alpha = y_\beta$, $x_\alpha > y_\beta$ *(trichotomy)*

3. Fuzzy Metric Space.

In the seuqal, (X,d) = X will denote a metric space.

As is well known, a fuzzy distance function $\widetilde{d}$ from $[\widetilde{P}(X)]^2$ to $\widetilde{P}(\mathbb{R}^+)$ is defined by for all fuzzy sets A and B in X

$$\mu_{\tilde{d}(A,B)}(\delta) = \bigvee_{\delta=d(u,v)} (\mu_A(u) \wedge \mu_B(v)) \qquad for \quad all \quad \delta \in \mathbb{R}^+$$

([ ]). If A and B are fuzzy points $x\alpha$ and $y\beta$, respectively, then $\tilde{d}(x_\alpha, y_\beta)$ is the fuzzy point $d(x,y)_{\alpha\wedge\beta}$ in $\mathbb{R}^+$.

Definition 3.1. ([1]) A function $\tilde{d}$ from $[F_p(X)]^2$ into $\tilde{P}(\mathbb{R}^2)$ defined by

$$\tilde{d}(x_\alpha, y_\beta) = d(x,y)_{\alpha\wedge\beta} \quad for \quad all \quad (x_\alpha, y_\beta) \in [F_p(X)]^2$$

is called the **induced metric** on X. The pair $(X,\tilde{d})$ is called the **induced fuzzy metirc space.**

Proposition 3.2. *Let $(X,\tilde{d})$ be an induced fuzzy metric space. Then for all x_α, y_β, $z_\gamma \in F_p(X)$ the followings hold;*

(1) $\tilde{d}(x_\alpha, y_\beta) \in F_p(\mathbb{R}^+)$.

(2) $\tilde{d}(x_\alpha, y_\beta) = 0_{\alpha\wedge\beta}$ if and only if $x = y$.

(3) $\tilde{d}(x_\alpha, y_\beta) = \tilde{d}(y_\beta, x_\alpha)$. *(symmetry)*

(4) Except z_γ with $x \le z \le y$ (or $y \le z \le x$) and $0 < \gamma < \alpha \wedge \beta$,

we have $\tilde{d}(x_\alpha, y_\beta) \le \tilde{d}(x_\alpha, z_\gamma) + \tilde{d}(z_\gamma, y_\beta)$ *(conditional triangle inequality)*

Definition 3.3. Let $x \in X$ and $r > 0$ be given. The fuzzy set $B(x_\alpha\ ;\ r_\alpha)$ defined by

$$B(x_\alpha, r_\alpha) = \bigcup \{y_\beta \in F_p(X) | \tilde{d}(x_\alpha, y_\beta) << r_\alpha\}$$

is called the fuzzy open ball with center x_α and radius r_α or the fuzzy r_α-neighborhood of x_α.

Remark. In the previous definition, consider, in general,

$$B(x_\alpha; r_\gamma) = \bigcup \{y_\beta \in F_p(X) | \tilde{d}(x_\alpha, y_\beta) << r_\gamma\}.$$

If $\alpha < \gamma$, then $B(x_\alpha\ ;\ r_\gamma)$ is ordinary open ball $B(x;\gamma)$. If $\alpha > \gamma$, then $B(x_\alpha\ ;\ r_\gamma)$ and $B(x_\gamma\ ;\ \gamma_\gamma)$ are equal. In these reasons, the values of center and radius of a fuzzy open ball should be equal.

Definition 3.4. ([1]) A fuzzy set A in X is said to be **fuzzy open** if for every $x \in \mathcal{S}(A)$ and for every $0 < \lambda < \mu_A(x)$ there exists an $\epsilon > 0$ such that $B(x_\lambda\ ;\ \epsilon_\lambda) \subset A$.

Proposition 3.5. *Every fuzzy open ball is fuzzy open.*

Porposition 3.6. *A fuzzy set A in X is fuzzy open if and only if if it is the union of fuzzy open balls.*

Theorem 3.7. *the family $\mathcal{T}$ of all fuzzy sets in X satisfies the followings;*

(OS 1) For each $\alpha \in I$, $X_\alpha \in \mathcal{T}$, where X_α is the fuzzy set in X which is characterized by $\mu_{X_\alpha}(x) = \alpha$ for all $x \in X$.

(OS 2) If $\{U_j \mid j \in J\} \subset \mathcal{T}$, then $\bigcup_{j \in J} U_j \in \mathcal{T}$.

(OS 3) If $U, V \in \mathcal{T}$, then $U \cap V \in \mathcal{T}$.

Definition 3.8. The family $\mathcal{T}$ in the Theorem 3.7 is called the **fuzzy metic topology** or **fuzzy topology induced by the metric** and the pair (X, $\mathcal{T}$) the **fuzzy metric topological space** or **fuzzy topological space induced by the metric**.

Definition 3.9. ([4]) A fuzzy set A is X is said to be **fuzzy closed** if its complement A^c is fuzzy open.

Proposition 3.10. *In a fuzzy metric topological space (X, $\mathcal{T}$), we have the followings;*

(1) If U is fuzzy open, then $S(U)$ is open.

(2) If F is fuzzy closed, then $S(F)$ is closed.

Theorem 3.11. *The family $\mathfrak{F}$ of all fuzzy closed sets in X satisfies the followings;*

(CS 1) For each $\alpha \in I$, $X_\alpha \in \mathfrak{F}$.

(CS 2) If $\{F_j \mid j \in J\} \subset \mathfrak{F}$, then $\bigcap_{j \in J} F_j \in \mathfrak{F}$.

(Cs 3) If $F_1, F_2 \in \mathfrak{F}$, then $F_1 \cup F_2 \in \mathfrak{F}$.

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