Limit Properties in the Fuzzy Real Line

Jeong-Yeol Choi and Ju-Ran Moon
Department of Mathematics, Wonkwang University

ABSTRACT

In this paper, we introduce the notion of limit in a usual fuzzy real function and investigate some of its properties.

1. Introduction

In [3], we had some properties for limits of usual fuzzy real sequences. In this paper, we introduce the notions of (usual fuzzy) curve, (fuzzy) connectedness and (fuzzy domain). From these, we define the limit of a certain usual fuzzy real function and have some properties for limits of such functions.

2. Limit of a Usual Fuzzy Real Sequence

Definition 2.1. ([2]) *The system* $\mathbf{R}$ *of all real numbers together with its usual fuzzy metric* $\tilde{d}$ *and its usual fuzzy topology is called the fuzzy real line.*

Definition 2.2. ([3]) *A function* s *from the set* $\mathbf{N}$ *of all natural numbers into the set* $F_p(\mathbf{R})$ *of all fuzzy points in* $\mathbf{R}$ *is called a usual fuzzy real sequence. The sequence* s *will be denoted by*

$$< x^{(1)}_{\alpha_1}, x^{(2)}_{\alpha_2}, \cdots, x^{(n)}_{\alpha_n} \cdots > \quad or \quad < x^{(n)}_{\alpha_n} >,$$

where the n*-th term* $s(n)$ *of* s *is the fuzzy point* $x^{(n)}_{\alpha_n}$ *with the support* $x^{(n)}$ *and the value* $\alpha_n \in (0, 1]$.

Definition 2.3. ([3]) *A usual fuzzy real sequence* $< x_{\alpha_n}^{(n)} >$ *is said to converge to a fuzzy point* $x_\alpha \in F_p(\mathbf{R})$*, denoted by* $x_{\alpha_n}^{(n)} \to x_\alpha$ *or* $\lim_{n\to\infty} x_{\alpha_n}^{(n)} = x_\alpha$ *if* $x^{(n)} \to x$ *and* $\alpha_n \to \alpha$*, i.e., for any given* $\epsilon > 0$ *there exists an* $N \in \mathbf{N}$ *such that* $n > N$ *implies* $|x^{(n)} - x| < \epsilon$ *and* $|\alpha_n - \alpha| < \epsilon$*. In this case,* x_α *is called the limit of* $< x_{\alpha_n}^{(n)} >$.

Proposition 2.4. *Let* $< x_{\alpha_n}^{(n)} >$ *be a usual fuzzy real sequence. Then, the followings are equivalent.*

(i) $< x_{\alpha_n}^{(n)} >$ *converges to* $x_\alpha \in F_p(\mathbf{R})$.

(ii) $\tilde{d}(< x_{\alpha_n}^{(n)}, x_\alpha) >$ *converges to* 0_α.

(iii) For each $U \in \mathcal{T}$ *with* $\mu_U(x) > \alpha$*. (In case* $\alpha = 1$*,* U *satisfies* $(x-\epsilon, x+\epsilon) \subset U$ *for some* $\epsilon > 0$*.)*

There exists an $N \in \mathbf{N}$ *such that* $n > N$ *implies* $x_{\alpha_n}^{(n)} \in U$*, and for every* $V \in \mathcal{T}$ *with* $\mu_V(x) < \alpha$ *there exists an* $n > n_0$ *such that* $< x_{\alpha_n}^{(n)} > \notin V$ *for all* $n_0 \in \mathbf{N}$.

Remark. *The condition for such* $V \in \mathcal{T}$ *in the proposition 2.4 (iii) is required bacause if there is no such condition, then for each* β *with* $\beta > \alpha$*.* x_β *is also a limit of* $< x_{\alpha_n}^{(n)} >$.

3. Limit of a Usual Fuzzy Real Function

Definition 3.1. *A function* f *from a subset* D *of* $F_p(\mathbf{R})$ *into* $F_p(\mathbf{R})$ *is called a usual fuzzy real function.*

Since we may identify a fuzzy point x_α in $\mathbf{R}$ with the point (x, α) in the plane, it is natural to define the following.

Definition 3.2. *A subset* C *of* $F_p(\mathbf{R})$ *is called a (usual fuzzy) curve if the subset* $\{(x,\alpha)|x_\alpha \in C\}$ *of* $\mathbf{R}^2$ *is a curve, i.e., there is a continuous function* c *from an interval* $[a, b]$ *in* $\mathbf{R}$ *onto* $\{(x,\alpha)|x_\alpha \in C\}$ *in* $\mathbf{R}^2$*. In this case, we say that* C *is a curve from the fuzzy point* $c(a)$ *to the fuzzy point* $c(b)$ *or* $c(a)$ *and* $c(b)$ *are joined by* C.

Definition 3.3. *A subset* D *of* $F_p(\mathbf{R})$ *is said to be (fuzzily) connected if every pair of fuzzy points* x_α *and* y_β *in* D *can be joined by a curve that lies entirely in* D.

Note that for a fuzzy set D_f in $\mathbf{R}$ we have

$$D_f = \bigcup_{x \in S(D_f)} \left(\bigcup_{0<\lambda\leq\mu_{D_f}(x)} x_\lambda \right),$$

where $S(D_f)$ is the support of D_f ([5]). In this reason, we may identify a fuzzy set D_f in $\mathbf{R}$ with the subset

$$D = \{x_\lambda | x \in S(D_f),\ \ 0 < \lambda \leq \mu_{D_f}(x)\}$$

of $F_p(\mathbf{R})$. In view of this, we have the following definition.

Definition 3.4. *A subset D of $F_p(\mathbf{R})$ is called a (fuzzy) domain or (fuzzy) open region if it is connected and there is a fuzzy open set D_f such that*

$$D = \{x_\lambda |\ x \in S(D_f),\ \ 0 < \lambda \leq \mu_{D_f}(x)\}.$$

In $\mathbf{R}^2$, we say that (x, y) approaches (a, b), written $(x, y) \to (a, b)$ if x approaches a and y approaches b. In this reason, it is natural to say that x_λ approaches a_α, written $x_\lambda \to a_\alpha$ if $x \to a$ and $\lambda \to \alpha$.

Definition 3.5. *Let D be a (fuzzy) domain and let f be a usual fuzzy real function from D into $F_p(\mathbf{R})$. We say that f has the limit $\ell_\beta \in F_p(\mathbf{R})$ as x_λ approaches a_α, written $\lim_{x_\lambda \to a_\alpha} f(x_\lambda) = \ell_\beta$ or $f(x_\lambda) \to \ell_\beta$ as $x_\lambda \to a_\alpha$, provided that for any given $\epsilon > 0$ there is a $\delta > 0$ such that $|x - a| < \delta$, $|\lambda - \beta| < \delta$ and $x_\lambda \in D - \{a_\alpha\}$ imply $|f_s(x_\lambda) - \ell| < \epsilon$ and $|f_v(x_\lambda) - \beta| < \epsilon$, where $f_s(x_\lambda)$ and $f_v(x_\lambda)$ are the support and the value of the fuzzy point $f(x_\lambda)$, respectively.*

Proposition 3.6. *Let D be a domain and let $f : D \to F_p(\mathbf{R})$. Then, the followings are equivalent.*

(i) $f(x_\lambda) \to \ell_\beta$ as $x_\lambda \to a_\alpha$.

(ii) For every usual fuzzy real sequence $< x^{(n)}_{\alpha_n} >$ in $D - \{a_\alpha\}$ with $x^{(n)}_{\alpha_n} \to a_\alpha$, $f(x^{(n)}_{\alpha_n}) >$ converges to ℓ_β.

(iii) If $U \in \mathcal{T}$ with $\mu_U(\ell) > \beta$, then there is a $W \in \mathcal{T}$ such that $x_\lambda \in W$ and $x_\lambda \in D$ imply $f(x_\lambda) \in U$ and if $V \in \mathcal{T}$ with $\mu_V(\ell) < \beta$, then for every $\delta > 0$ there is an $x_\lambda \in D - \{a_\alpha$ such that $|x - a| < \delta$ and $|\lambda - \beta| < \delta$ but $f(x_\lambda) \notin V$.

Proposition 3.7. *Let D be a domain and let $f, g : D \to F_p(\mathbf{R})$. If* $\lim_{x_\lambda \to a_\alpha} f(x_\lambda) = s_\beta$ and $\lim_{x_\lambda \to a_\alpha} g(x_\lambda) = t_\gamma$

(1) $\lim_{x_\lambda \to a_\alpha} [f(x_\lambda + g(x_\lambda)] = s_\beta + t_\gamma.$

(2) $\lim_{x_\lambda \to a_\alpha} [f(x_\lambda - g(x_\lambda)] = s_\beta - t_\gamma.$

(3) For any given ℓ_θ, $\lim_{x_\lambda \to a_\alpha} [\ell_\theta f(x_\lambda)] = \ell_\theta s_\beta = (\ell s)_{\theta \wedge \beta}.$

(4) $\lim_{x_\lambda \to a_\alpha} [f(x_\lambda) g(x_\lambda)] = s_\beta t_\gamma = (st)_{\beta \wedge \gamma}.$

(5) Let $0 \notin S(g(D))$ *and let* $t \neq 0$.
Then, $\lim_{x_\lambda \to a_\alpha} [f(x_\lambda)/g(x_\lambda)] = s_\beta / t_\gamma = (s/t)_{\beta \wedge \gamma}.$

Remark. *Let D be a domain and $f : D \to F_p(\mathbf{R})$. We define that F is (fuzzy) continuous at $a_\alpha \in D$ if* $\lim_{x_\lambda \to a_\alpha} f(x_\lambda) = f(a_\alpha)$*. Then, we obtain similar results to the propositions 3.6 and 3.7.*

References

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