Laws of Large Numbers for Partial Sum Processes of Fuzzy Random Variables

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Abstract

We prove a uniform strong law of large numbers for sequences of fuzzy random variables.

1. Introduction

Kruse [7] proved a strong law of large numbers for independent and identically distributed fuzzy random variables and Klement, Puri and Ralescu [6] obtained a strong law of large numbers and some other limit theorems. Miyakoshi and Shimbo [9] obtained a strong law of large numbers for independent fuzzy random variables and they [10] also generalized Birkhoff's ergodic theorem to fuzzy random variables. The present paper is concerned with the investigation of a limit theorem usually refered to as the uniform strong law of large numbers in the presence of fuzzyness. Our results generalize that of Bass and Pyke[2].

2. Preliminaries

Assume that R is the set of real numbers. Let $C(R) = \{A \subset R : A \text{ compact}\}$ and $K(R) = \{A \in C(R) : A \text{ convex}\}$. The space $C(R)(K(R))$ has a linear structure induced by the operations $A + B = \{a + b : a \in A, \quad b \in B\}$ and $\lambda A = \{\lambda a : a \in A\}$ for $A, B \in C(R)(K(R))$, $\lambda \in R$. The Hausdorff distance between two sets A, B of $C(R)$ is defined as

$$d_H(A, B) = \max\{\sup_{a\in A} \inf_{b\in B} \|a - b\|, \sup_{b\in B} \inf_{a\in A} \|a - b\|\}.$$

An extension of $C(R)$ is obtained by defining the space $\mathcal{F}(R)$ of fuzzy set $u : R \to [0, 1]$ satisfying

(1) u is upper semi-continuous,
(2) $\{x \in R : u(x) \geq \alpha\}$ is compact for each α,
(3) $\{x \in R : u(x) = 1\} \neq \emptyset$.

Furthermore, a subspace of $\mathcal{F}(R)$ denoted by $\mathcal{F}_c(R)$, is defined by requiring

(4) $\{x \in R : u(x) \geq \alpha\}$ to be compact and convex for each $\alpha > 0$.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

For each such fuzzy set u, we denote by $L_\alpha(u) = \{x \in R : u(x) \geq \alpha\}$, $\alpha \in [0,1]$ its α-level set. The space $\mathcal{F}(R)$ extends $K(R)$ in the sense that for each $A \in K(R)$, its characteristic function $\chi_A \in \mathcal{F}(R)$. Now we define generalized metric on $\mathcal{F}(R)$

$$d(u,v) = \int_0^1 d_H(L_\alpha(u), L_\alpha(v))d\alpha.$$

Note that $(\mathcal{F}(R), d)$ is complete(Puri and Ralescu [12]) and separable(Klement, Puri and Ralescu[6]). The norm $\|u\|$ of a fuzzy set $u \in \mathcal{F}(R)$ is defined by $\|u\| = d(u, I_{\{0\}}) = \sup \|L_\alpha(u)\|$.

If (Ω, P) is a probability space, a random interval, as a generalization of a random variable, is defined as a Borel measurable function $X : \Omega \to (C(R), d_H)$. The expected value EX(Aumann[1]; Chow and Teicher[3]) is defined by $EX = \{E\phi | \phi : \Omega \to R, E|\phi| < \infty, \quad \phi(\omega) \in X(\omega) \text{a.e.}\}$. Note that if $E\|X\| < \infty$ where $\|X\|(\omega) = \sup_{a \in X(\omega)} |a|$, then $E(X) \in K(R)$. A fuzzy random variable is a Borel measurable function $X : \Omega \to (\mathcal{F}(R), d)$ such that every $\alpha \in (0,1]$, the random variable $X_\alpha : \Omega \to C(R)$ defined by $X_\alpha(u) = \{x \in R : X(\omega)(x) \geq \alpha\}$ is a compact set in R. In defining the expected value of a fuzzy random variable X, theorem 3.1 in Puri and Ralescu [13] assures that $L_\alpha(E(coX)) = EL_\alpha(coX), 0 < \alpha \leq 1$. It follows that $E\|L_\alpha(X)\| < \infty$ implies $EX \in \mathcal{F}_c(R)$.

3. Main Theorems

Let $J = \{1, 2, \cdots, \}^d$ and let $\{X_j : j \in J\}$ be a family of independent identically distributed random intervals (or fuzzy random variables) with common expected value EX. Let $C \subset [0, \infty)^d$ with d positive integer be Borel measurable and let $|C|$ denote the Lebesgue measure of C. Define $S(C) = \sum_{j \in C} X_j$ to be the partial sum of random intervals (or fuzzy random variables).

Given a set $B \subset [0,1]^d$, let $nB = \{nx : x \in B\}$ and $B(\delta) = \{x : \rho(x, \partial B) < \delta\}$ be the δ-annulus of ∂B, where $\rho(\cdot,\cdot)$ is the Euclidean distance and ∂B is the boundary of B. Let $|B|$ denote the Lebesgue measure of B. Let $\mathcal{A}$ be a family of Borel measurable subsets of $[0,1]^d$. Define $r(\delta) = \sup_{A \in \mathcal{A}} |A(\delta)|$. We say that $\mathcal{A}$ satisfies the smooth boundary condition(SBC) when $r(\delta) \to 0$ as $n \to \infty$. Under SBC on $\mathcal{A}$, we have the following uniform strong law of large numbers for fuzzy random variables:

Theorem 1. *Let $\{X_j : j \in J\}$ be a family of independent identically distributed random intervals with $EX_j = [\mu_1, \mu_2]$. Suppose $\mathcal{A}$ is a collection of Lebesgue measurable subsets of $[0,1]^d$ such that $r(\delta) = \sup_{A \in \mathcal{A}} |A(\delta)| \to 0$ as $\delta \to 0$, then*

$$\sup_{A \in \mathcal{A}} d_H\left(\frac{S(nA)}{n^d}, |A|[\mu_1, \mu_2]\right) \to 0 \quad a.s.$$

as $n \to \infty$ where d_H is the Hausdorff metric on subsets of $[0,1]^d$.

Theorem 2. *Let $\{X_j : j \in J\}$ be a family of independent identically distributed fuzzy random variables with common expected value EX. Suppose $\mathcal{A}$ is a collection of Lebesgue measurable subsets of $[0,1]^d$ such that $r(\delta) = \sup_{A\in\mathcal{A}} |A(\delta)| \to 0$ as $\delta \to 0$, then*

$$\sup_{A\in\mathcal{A}} d\left(\frac{S(nA)}{n^d}, |A|EX\right) \to 0 \quad a.s.$$

as $n \to \infty$ where d is the generalized metric induced from the Hausdorff metric d_H on subsets of $[0,1]^d$.

Proof of theorem 2. For each $\alpha \in [0,1]$, the sequence $\{X_{j\alpha}\}$ satisfies theorem 1. To prove theorem 2 what we have to show is the following:

$$\sup_{A\in\mathcal{A}} \int_0^1 d_H\left(L_\alpha\left(\frac{S(nA)}{n^d}\right), |A|L_\alpha(EX)\right) d\alpha \to 0 \quad a.s.$$

as $n \to \infty$. To do this first we need to show that for a fixed rectangle A.

$$\int_0^1 d_H\left(L_\alpha\left(\frac{S(nA)}{n^d}\right), \| A|L_\alpha(EX)\right) d\alpha \to 0 \quad a.s.$$

as $n \to \infty$. Under the set-ups as in section 3, Klement, Puri and Ralescu[6]'s strong laws implies, $n^{-d}S(n(0,\mathbf{x}]) = \frac{\sharp(J\cap n(0,\mathbf{x}])}{n^d} \cdot \frac{S(n(0,\mathbf{x}])}{\sharp(J\cap n(0,\mathbf{x}])} \to |(0,\mathbf{x}]|EX_1$. If A can be obtained by a finite number of unions and differences of rectangles of the form $(0,\mathbf{x}]$, then by linearity we have $n^{-d}S(nA) \to |A|EX$ a.s. Now let $\nu_\alpha = \max\{E|s_{1\alpha}|, E|s_{2\alpha}|\}$ and $T_{i\alpha}(A) = \sum_{j\in A} |s^j_{i\alpha}|$, for $i = 1,2$ and $\alpha \in (0,1]$. Then, for m fixed

$$\begin{aligned}
\limsup_{n\to\infty}\sup_{A\in\mathcal{A}} d(n^{-d}S(nA), |A|EX) \le\ & \limsup_{n\to\infty}\sup_{A\in\mathcal{A}} n^{-d}d(S(nA), S(nR_m^-(A))) \\
&+ \limsup_{n\to\infty}\sup_{A\in\mathcal{A}} d(n^{-d}S(nR_m^-(A)), |R_m^-(A)|EX) \\
&+ \limsup_{n\to\infty}\sup_{A\in\mathcal{A}} d(|A|EX, |R_m^-(A)|EX) \\
=\ & I_1 + I_2 + I_3.
\end{aligned}$$

Firstly, with some calculation we have

$$I_3 \le \|suppEX\| r(d^{1/2}m)$$

Secondly

$$\begin{aligned}
I_2 &= \limsup_{n\to\infty, A\in\mathcal{A}} d(n^{-d}S(nR_m^-(A)), |R_m^-(A)|EX) \\
&\le \limsup_{n\to\infty, B\in\mathcal{R}_m^-} d(n^{-d}S(nB), |B|EX) \\
&\le \limsup_{n\to\infty} \max_{B\in\mathcal{R}_m^-} d(n^{-d}S(nB), |B|EX) \\
&= \limsup_{n\to\infty} \max_{B\in\mathcal{R}_m^-} \int_0^1 d_H(L_\alpha(n^{-d}S(nB)), L_\alpha(|B|EX))d\alpha \\
&\le \int_0^1 \limsup_{n\to\infty} \max_{B\in\mathcal{R}_m^-} d_H(L_\alpha(n^{-d}S(nB)), L_\alpha(|B|EX))d\alpha \\
&= 0 \quad \text{a.s.}
\end{aligned}$$

where we used the fact that $\sharp\mathcal{R}_m^- < \infty$ and every set $B \in \mathcal{R}_m^-$ can be obtained by a finite number of unions and differences of rectangles of the form $(0, \mathbf{x}]$.

Now

$$d(S(nA), S(nR_m^-(A))) = \int_0^1 \max\{\sum_{j\in(nR_m^+(A)\setminus nR_m^-(A))} |s_{1\alpha}^j|, \sum_{j\in(nR_m^+(A)\setminus nR_m^-(A))} |s_{2\alpha}^j|\}d\alpha.$$

Therefore

$$\begin{aligned} I_1 &\le \limsup_{n\to\infty, A\in\mathcal{A}} n^{-d}\int_0^1 \max\{\sum_{j\in(nR_m^+(A)\setminus nR_m^-(A))} |s_{1\alpha}^j|, \sum_{j\in(nR_m^+(A)\setminus nR_m^-(A))} |s_{2\alpha}^j|\}d\alpha \\ &\le \int_0^1 \limsup_{n\to\infty, A\in\mathcal{A}} n^{-d}\max\{\sum_{j\in(nR_m^+(A)\setminus nR_m^-(A))} |s_{1\alpha}^j|, \sum_{j\in(nR_m^+(A)\setminus nR_m^-(A))} |s_{2\alpha}^j|\}d\alpha \\ &\le \|supptX\| r(d^{1/2}/m) \quad \text{a.s.} \end{aligned}$$

where we used the fact that $\sharp\mathcal{R}_m^{\triangle}$ was finite. Hence summing up, we have

$$\limsup_{n\to\infty\, A\in\mathcal{A}} d(n^{-d}S(nA), |A|EX) \le 2\|suppX\| r(d^{1/2}/m) \quad \text{a.s.}$$

Letting $m \to \infty$ concludes the proof.

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