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On the Stabilization of Linear Discrete Time Systems Subject to Input Saturation

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Abstract In this paper, a linear discrete time system subject to the input saturation is shown to be exponentially stabilizable on any compact subset of the constrained asymptotically stabilizable set by a linear periodic variable structure controller. We also establish that any neutrally stable system subject to the input saturation can be globally asymptotically stabilizable via linear feedback.

Keyword Constrained stabilization; Constrained asymptotically stabilizable set; Linear periodic variable structure control; Linear discrete time systems

1. Introduction

It is shown in [3], [4] that an input constrained linear discrete time system can be globally asymptotically stabilizable iff all its poles are located inside or on the unit circle. In [4], a nonlinear globally stabilizing control law for marginally stable systems is constructed. However, Yang [5] showed a negative result that the constrained global stabilization of such systems is in general impossible through linear feedback. Nevertheless, Lin and Saberi [2], [1] showed that there exists a linear control that exponentially stabilizes a marginally stable system on any bounded subset of the state space. These constrained stabilization results focused on marginally stable systems and the constrained stabilization of unstable systems has not been addressed yet.

In this paper, we first show that any neutrally stable systems can be globally asymptotically stabilized through linear feedback in the presence of the input saturation. Then we address the constrained stabilization of unstable systems. For stable and marginally stable systems, the desired region for the stabilization was the entire state space. However, as mentioned above, the constrained global stabilization is impossible for unstable systems. Hence, for unstable systems, it is important to know the region over which the constrained stabilization is possible. The largest possible region for the constrained stabilization is the constrained asymptotically stabilizable set. Hence, we first establish the properties and structure of the constrained asymptotically stabilizable set. We then show that any unstable systems can be exponentially stabilized by a linear periodic variable structure controller on any compact subset of the constrained asymptotically stabilizable set.

2. Main Results

Consider the system

$$x(k+1) = Ax(k) + B\sigma(u(k)), \qquad x(0) = x_0, \quad (1)$$
$$y(k) = Cx(k),$$

where $x(k) \in \mathbf{R}^n$, $u(k) \in \mathbf{R}^m$, $y(k) \in \mathbf{R}^l$, and

$$\sigma(u) := \begin{cases} -u_{lim} & \text{if } u < -u_{lim} \\ u & \text{if } -u_{lim} \le u \le u_{lim} \\ u_{lim} & \text{if } u > u_{lim} \end{cases}.$$

Throughout the paper, the following two assumptions are adopted. Firstly, the stabilizability of (A, B) is assumed that is necessary for the constrained stabilizability. Secondly, we assume without loss of generality that all the eigenvalues are unstable. Under this assumption, the first assumption reduces to the controllability of (A, B) and the constrained stabilizable set to the constrained null controllable set.

We need the following properties of the saturation function σ in the sequel.

Fact 2.1 [4]: $s^T\sigma(s) > 0$ if $s \neq 0$, and $s^T\sigma(s) \geq \sigma(s)^T\sigma(s)$.

We first show that, similar to the continuous time case [4], the global asymptotic stabilization of the saturated system is possible through linear feedback if System (1) is neutrally stable: i.e.

Jordan blocks associated with eigenvalues outside the open unit disk are simple and all the other eigenvalues are in the interior of the open unit disk. Frist notice that there exists an invertible matrix T for which $T^{-1}AT$ is unitary such that $(T^{-1}AT)^T T^{-1}AT = I$. Let $\hat{A} := T^{-1}AT$ and $\hat{B} := T^{-1}B$. Then define

$$u(k) = -\sigma(\kappa \hat{B}^T \hat{A} T^{-1} x),$$

where $\kappa > 0$ is to be chosen later. Consider the following Lyapunov function candidate:

$$V(x) = (T^{-1}x)^T T^{-1} x.$$

Clearly, V is continuous, positive definite, and radially unbounded. Moreover, it holds that

$$\begin{aligned}
&\Delta V(x(k+1)) = V(x(k+1)) - V(x(k)) \\
&= (T^{-1}x(k))^T \hat{A}^T \hat{A} T^{-1} x(k) + 2(T^{-1}x(k))^T \hat{A}^T B u(k) \\
&\quad + u(k) \hat{B}^T \hat{B} u(k) - (T^{-1}x(k))^T T^{-1} x(k) \\
&= (T^{-1}x(k))^T (\hat{A}^T \hat{A} - I) T^{-1} x(k) \\
&\quad - \frac{2}{\kappa} (T^{-1}x(k))^T \hat{A}^T \hat{B} \kappa \sigma(\kappa \hat{B}^T \hat{A} T^{-1} x(k)) \\
&\quad + \sigma(\kappa \hat{B}^T \hat{A} T^{-1} x(k))^T \hat{B}^T \hat{B} \sigma(\kappa \hat{B}^T \hat{A} T^{-1} x(k)) \\
&= -\frac{2}{\kappa} (T^{-1}x(k))^T \hat{A}^T \hat{B} \kappa \sigma(\kappa \hat{B}^T \hat{A} T^{-1} x(k)) \\
&\quad + \sigma(\kappa \hat{B}^T \hat{A} T^{-1} x(k))^T \hat{B}^T \hat{B} \sigma(\kappa \hat{B}^T \hat{A} T^{-1} x(k)).
\end{aligned}$$

Now choose $\kappa > 0$ such that

$$\hat{B}^T \hat{B} - \frac{2}{\kappa} I < 0.$$

Then from Fact 2.1, $\Delta V(\cdot) \leq 0$. Suppose $\Delta V \equiv 0$ along a trajectory $x(k)$. Then it must hold that, along the trajectory, $\hat{B}^T \hat{A} T^{-1} x(k) \equiv 0$ and, in turn, $u(k) \equiv 0$. Since it holds along the trajectory that

$$T^{-1}x(k+1) = \hat{A} T^{-1} x(k),$$

it follows that

$$0 = \hat{B}^T \hat{A} T^{-1} x(k+1) = \hat{B}^T \hat{A}^2 T^{-1} x(k).$$

Similarly, it holds along the trajectory that

$$\hat{B}^T \hat{A}^i T^{-1} x(k) = 0$$

for all i. This implies

$$0 = \begin{bmatrix} \hat{B}^T \hat{A} \\ \vdots \\ \hat{B}^T \hat{A}^n \end{bmatrix} T^{-1} x(k)$$

$$= \begin{bmatrix} \hat{B}^T (\hat{A}^T)^{n-1} \\ \vdots \\ \hat{B}^T \end{bmatrix} \hat{A}^n T^{-1} x(k).$$

Hence, from the controllability of (A, B), it follows that $x(k) \equiv 0$. Then the global asymptotic stability follows from LaSalle's Invariance Principle.

We now consider the exponential stabilization of unstable systems subject to the input saturation on any compact subset of the constrained asymptotically stabilizable set via linear periodic variable structure feedback. Let

$$U := \{u \in \mathbf{R}^m : -u_{sat} \leq u \leq u_{sat}\}.$$

Clearly, U is a rectangle in $\mathbf{R}^m$ that is compact, convex and symmetric with respect to the origin.

We now define the constrained N step null controllable set as:

$$C_N := \{x_0 \in \mathbf{R}^n \,|\, \exists u(k) \in U, x(N) = 0\}$$

and the constrained asymptotically null controllable set as:

$$C_\infty := \left\{ x_0 \in \mathbf{R}^n \,\middle|\, \exists u(k) \in U, \lim_{k \to \infty} x(k) = 0 \right\}.$$

It is trivial to show that C_N and C_∞ have the following properties.

Fact 2.2: $C_N \subset C_\infty$, for all N.

Fact 2.3: $C_N \subset C_{N'}$ if $N \leq N'$.

Theorem 2.1: C_N, $0 \leq N \leq \infty$, is convex and symmetric with respect to the origin.

Proof: Let $x_0^1, x_0^2 \in C_N$. Suppose $u^1(k), u^2(k) \in U$ are input sequences that drive x_0^1, x_0^2 to the origin, respectively. Then, for $0 \leq a \leq 1$, the input sequence $au^1(k) + (1-a)u^2(k) \in U$ drives $ax_0^1 + (1-a)x_0^2$ to the origin. Hence, $ax_0^1 + (1-a)x_0^2 \in C_N$ and the convexity of C_N follows.

Let $x_0 \in C_N$. Suppose $u(k) \in U$ is an input sequence that drives x_0 to the origin. Then the input sequence $-u(k) \in U$ drives $-x_0$ to the origin. Hence, $-x_0 \in C_N$ and the symmetry of C_N follows. □

We further explore the structure of C_N. For this, first notice that

$$x(k) = A^k x_0 + \sum_{i=0}^{k-1} A^{k-i-1} B u(i).$$

Then $x_0 \in C_N$ iff there exists $u(k) \in U$ such that

$$0 = A^N x_0 + \sum_{i=0}^{N-1} A^{N-i-1} B u(i)$$

or

$$x_0 = -\sum_{i=0}^{N-1} A^{-i-1} Bu(i).$$

Hence, C_N is simply the set of all points that are linear combinations of $A^{-i-1}B$, $i = 0, \cdots, N-1$ whose coefficients are in U. This implies C_{N+1} is the set of all points of the form $x_0 = \bar{x}_0 + A^{-N-1}Bu$ such that $\bar{x}_0 \in C_N$ and $u \in U$. Hence, C_{N+1} is the polyhedron that can be represented as:

$$C_{N+1} = \cup_{\bar{x}_0 \in C_N} \{x_0 = \bar{x}_0 + A^{-N-1}bu : u \in U\}$$

$$= C_N \cup \left[\cup_{\bar{x}_0 \in \partial C_N} \{x_0 = \bar{x}_0 + A^{-N-1}bu : u \in U\} \right]. \tag{2}$$

Lemma 2.1: For $N \geq n$, there exists an open ball $B^n(0, r) \subset C_N$.

Proof: From the controllability of (A, B), $A^{-i-1}B$, $i = 0, \cdots, n-1$ are linearly independent. Moreover, U contains a nonempty interior that contains the origin. Hence, the lemma follows. □

Similarly, we can show the following lemma.

Lemma 2.2: $C_N \subset \overset{\circ}{C}_M$ for all $M \geq N + n$..

Theorem 2.2: $C_\infty = \cup_{0 \leq N < \infty} C_N = \lim_{N \to \infty} C_N$.

Proof: From Fact 2.2, it is trivial to show that $C_\infty \supset \cup_{0 \leq N < \infty} C_N$. From Lemma 2.1, for $N \geq n$, there exists an open ball $B^n(0, r) \subset C_N$. Hence, given $x_0 \in C_\infty$, there exists M such that there exists a input sequence in U for which $x(i) \in C_N$ for all $i \geq M$. This implies there exists $u(i) \in U$ such that $x(N + M) = 0$. Thus, $x_0 \in C_{N+M}$ and $C_\infty \subset \cup_{0 \leq N < \infty} C_N$. Hence, the first equality follows. The second equality is obvious from Fact 2.3. □

Clearly, C_N is closed for all N. However, from Theorem 2.2 and Lemma 2.2, it can be shown that C_∞ is open.

Corollary 2.1: C_∞ is open.

Proof: Suppose the contrary. Then there exists $x_0 \in \partial C_\infty$. From the proof of Theorem 2.2, for x_0, there exists L such that there exists an input sequence in U for which $x(L) = 0$. Then $x_0 \in C_L$. This is a contradiction from Lemma 2.2 and the corollary follows. □

We now show that for big enough N, C_N can arbitrarily closely approximate C_∞.

Theorem 2.3: If W is a compact set contained in C_∞, there exists N' such that $W \subset C_N$ for all $N \geq N'$.

Proof: From the proof of Theorem 2.2, for any $x_0 \in C_\infty$, there exists L such that there exists an input sequence in U for which $x(L) = 0$. Suppose there doesn't exist N' such that $W \subset C_N$ for all $N \geq N'$. Then there exists a sequence $\{z_i\}$ such that $z_i \in W$, $z_i \in C_i$ and $z_i \notin C_{i-1}$. Since W is compact, the sequence converges to a point $\bar{z} \in W$. However, from Lemma 2.2, $\bar{z} \notin C_N$ for any N and, thus, $\bar{z} \notin C_\infty$. This is a contradiction and the theorem follows. □

Let $\{x_i^N\}_{i=1}^{p_N}$ be the set of all vertices of C_N. For each i, x_i^N can be represented as

$$x_i^N = -\sum_{j=0}^{N-1} A^{-j-1} B u_i^N(j),$$

where $u_i^N(j) \in U$. Then the input $[u_i^N(0) \cdots u_i^N(N-1)]$ drives x_i^N to the origin in N steps. For any $\hat{x} \in C_N$, there exist nonnegative real numbers a_i's for which $\sum_{j=1}^{p_N} a_j \leq 1$ such that $\hat{x} = \sum_{j=1}^{p_N} a_j x_i^N$. Then $\sum_{j=1}^{p_N} a_j u_j^N(k) \in U$ for all $0 \leq k \leq N-1$ and the open loop control $[\sum_{j=1}^{p_N} a_j u_j^N(0) \cdots \sum_{j=1}^{p_N} a_j u_j^N(N-1)]$ drives $\hat{x}$ to the origin in N steps.

From Theorem 2.3, given a compact set W contained in C_∞, there exists $N \geq n$ such that $W \subset C_N$. We now construct an exponentially stabilizing linear periodic variable structure feedback control law on C_N. Suppose $x \in C_N$. Then there exists a face[1], S_j, of the polyhedron C_N such that x is contained in the polyhedral sector defined by the origin and S_j. Moreover, there exists a unique set of nonnegative real numbers a_{xi}^N's for which $\sum_{i=1}^n a_{xi}^N \leq 1$ such that $x = \sum_{i=1}^n a_{xi}^N x_i^N$, where $\{x_i^N\}$ is the set of all vertices of S_j. Now define

$$u_0(x) := \sum_{i=1}^n a_{xi}^N u_i^N(0)$$

$$= [u_1^N(0) \cdots u_n^N(0)][x_1^N \cdots x_n^N]^{-1} x.$$

For $x \in C_{N-1}$, define

$$u_1(x) := \sum_{i=1}^n a_{xi}^{N-1} u_i^{N-1}(0)$$

$$= [u_1^{N-1}(0) \cdots u_n^{N-1}(0)][x_1^{N-1} \cdots x_n^{N-1}]^{-1} x,$$

where $\{x_i^{N-1}\}$ is the set of all vertices of the face of C_{N-1} whose corresponding polyheral sector contains x and $\{a_{xi}^{N-1}\}$ is a unique set of nonnegative real numbers for which $\sum_{i=1}^n a_{xi}^{N-1} \leq 1$ such that $x = \sum_{i=1}^n a_{xi}^{N-1} x_i^{N-1}$. Now define $u_k(\cdot)$ for

[1] In this paper, a face is a simplice that is obtained by possibly dividing a face into simplices.

$k = 2, \cdots, n-1$, in similar way. For the sampling times greater than $n-1$, we define

$$u_{pn+q}(\cdot) = u_q(\cdot), \quad q = 0, \cdots, n-1, \quad p = 1, \cdots.$$

To this end, we have designed a linear periodic variable structure controller.

Now consider an initial condition $x_0 \in C_N$. Then, it holds that

$$x(1) = Ax_0 + B\sigma(u_0(x_0))$$

$$= A\sum_{i=1}^{n} a_{x_0 i}^N x_i^N + B\sum_{i=1}^{n} a_{x_0 i}^N u_i^N(0) \in C_{N-1}.$$

Hence, u_1 is well defined at $x(1)$ and, thus,

$$x(2) = Ax(1) + B\sigma(u_1(x(1)))$$

$$= A\sum_{i=1}^{n} a_{x(1)i}^{N-1} x_i^{N-1} + B\sum_{i=1}^{n} a_{x(1)i}^{N-1} u_i^{N-1}(0) \in C_{N-2}.$$

Similarly, $x(i) \in C_{N-i}$ for $k = 3, \cdots, n$. To this end, Fact 2.3 dictates that the proposed feedback control law is well-defined for all $x_0 \in C_N$ throughout the trajectory.

We now examine the stability of the proposed control law. For this, we need the follwong facts.

Fact 2.4: There exists $\beta \in [0, 1)$ such that $C_{N-n} \subset \beta^n C_N$.

Proof: The proof is trivial from Lemma 2.2. □

Fact 2.5: $x(pn+q) \in \beta^{pn} C_{N-q} \subset \beta^{pn} C_N$.

Proof: The proof is trivial from Fact 2.4 and the construction of the control. □

Let $r_{max} := \max_{x \in \partial C_N} |x|$ and $r_{min} := \min_{x \in \partial C_N} |x|$. Then, from Fact 2.5, it holds that $x_0, x(1), \cdots, x(n-1) \in \frac{|x_0|}{r_{min}} C_N$. This implies

$$|x_0|, |x(1)|, \cdots, |x(n-1)| \leq |x_0| \frac{r_{max}}{r_{min}}.$$

Similarly, it holds that $x(pn), \cdots, x((p+1)n-1) \in \beta^{pn} \frac{|x_0|}{r_{min}} C_N$ and, in turn,

$$|x(pn)|, \cdots, |x((p+1)n-1)| \leq \beta^{pn} |x_0| \frac{r_{max}}{r_{min}}.$$

Hence, for $x_0 \in C_N$, it holds that

$$|x(k)| \leq |x_0| \frac{r_{max}}{r_{min}\beta^n} \beta^k.$$

Hence, the closed loop system with the above contol is exponentially stable on C_N. To this end, we have the following Theorem.

Theorem 2.5: Given any compact subset W of C_∞, there exists N such that $W \subset C_N$ and System (1) can be exponentially stabilized on C_N by a linear periodic variable structure controller.

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