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# A POLE ASSIGNMENT METHOD IN A SPECIFIED DISK

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**Abstract** In this paper, a pole assignment problem in the unit disk for a linear discrete system is discussed. The analysis is based on the Luenberger's canonical form and Gershgorin's disk. The proposed method for pole assignment is convenient for a linear time invariant discrete system.

**Keywords** Luenberger's canonical form, Gershgorin's disk

## 1. INTRODUCTION

To get satisfactory performance in terms of speed, damping ratio and overshoot, etc. for linear time invariant systems, the poles locations for the closed loop system are constrained in a specified region. Usually, the problem can be solved by specifying a particular region of the left half complex plane for the continuous systems or in the unit circle for the discrete systems and by obtaining appropriate feedback control law which assign the closed loop system's poles in the specified region.

In this paper, the problem of pole assignment in a specified disk for linear discrete systems is addressed. Much research concerning this problem has been done and many methods are proposed in the last two, three decades. Bogachev and Grigorev [1] transformed the Lyapunov equation into other equation using linear fractional function and obtained the control law by solving the transformed equation. However, the control law is only determined by using the iteration of complicated process. Furuta and Kim [2] proposed a method for pole assignment in a specified disk by using the well known discrete Riccati equation. In their proposed algorithm, the state feedback law is determined by using the solution of a discrete Riccati equation which can be computed directly using the design specification parameters. Kim and Furuta [3] proposed a different pole assignment method adopting the linear fractional transformation. The design approach is firstly to transform the original system by linear fractional mapping, then secondly, pole assignment is performed for the transformed system by solving the Riccati equation, i.e. the state feedback law for the transformed system is obtained by assigning all the poles of the closed-loop system to the complex left half plane for continuous systems or to the unit circle, for discrete systems. Thirdly, the inverse mapping of the control law, i.e. the control law of the original system is derived. Among numerous works in analysis of pole assignment problem, some of them have derived extended Lyapunov equations that provide necessary and sufficient conditions for a given matrix to have all those eigenvalues in a specified disk [4]-[6]. In these works, the control law is derived via complicate programing. All the above methods use the Lyapunov or the Riccati equations but there is no method incorporating the matrix characteristics with the property of disk.

In this paper, a pole assignment method in a specified disk for linear discrete systems is introduced using the Luenberger's canonical form and Gershgorin's theorem. Firstly, in order to make clear the relationship between the feedback law and the original system the system is rearranged into a canonical system based on Luenberger's method. Secondly, using Gershgorin's theorem, we assign appropriate poles for the canonical system, then we obtain the feedback control law with closed-loop poles of the original system in a specified disk based on the relation between the original and the transformed systems.

## 2. PRELIMINERIES

In this section, some basic properties and theorems related with the pole assignment method in a specified disk for linear discrete systems are shown by using and modifying the well known results.

Let us consider the linear discrete system $\sum_D(A,B)$ :

$$x_{k+1} = Ax_k + Bu_k \tag{2.1a}$$

$$y_{k+1} = Cx_k \tag{2.1b}$$

where $x$ is an $n$-dimensional state vector, $u$ is an $m$-dimensional input vector, $y$ is a $p$-dimensional output vector and $A, B, C$ are constant matrices of appropriate dimensions. It is also assumed that the pair $(A, B)$ is controllable.

### 2.1 Problem statement

The problem to be addressed is to determine the state feedback

$$u_k = Fx_k \tag{2.2}$$

such that all the poles of the closed loop system (2.1) are located in the unit disk.

### 2.2 Basic theory

In this section, we introduce the basic theory and concepts which are the basis to the proposed method.

**[*Lemma 2.1*]** [7] (Luenberger's canonical form)

If an $m$ input, $p$ output, and $n$ dimensional linear multivariable system $(A, B)$ is controllable, then there exists a nonsingular matrix $T$ satisfying

$$\overline{A} = T^{-1}AT \ ; \ \overline{B} = T^{-1}B \tag{2.3}$$

where,

$$\overline{A} = \begin{bmatrix} \overline{A}_{11} & \cdots & \overline{A}_{1m} \\ \overline{A}_{21} & \cdots & \overline{A}_{2m} \\ \vdots & & \vdots \\ \overline{A}_{m1} & \cdots & \overline{A}_{mm} \end{bmatrix}, \quad \overline{B} = \begin{bmatrix} \overline{B}_1 \\ \overline{B}_2 \\ \vdots \\ \overline{B}_m \end{bmatrix},$$

$$\overline{A}_{ii} = \begin{bmatrix} 0 & & \\ \vdots & I_{\delta_i - 1} & \\ 0 & & \\ -\alpha_{i\rho_{i-1}} & \cdots & -\alpha_{i(\rho_i - 1)} \end{bmatrix} \in R^{\delta_i \times \delta_i},$$

$$\overline{A}_{ij} = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \\ -\alpha_{i\rho_{j-1}} & & -\alpha_{i\rho_j - 1} \end{bmatrix} \in R^{\delta_i \times \delta_i},$$

$$\overline{B}_i = \begin{bmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ \vdots & & 0 & \cdots & 0 \\ 0 & \cdots & 1 & \beta_{(i+1)i} \cdots & \beta_{mi} \end{bmatrix} \in R^{\delta_i \times m}$$

with

$$\rho_i = \sum_{j=1}^{i} \delta_j, (i = 1, \cdots, m;\ j = 1, \cdots, m),\ \rho_o = 0$$

and $\delta_i$ are called the controllability indexes. ■

Let $B = [\, b_1, b_2, b_3, \cdots, b_m]$, then from the controllability matrix

$$\{\, b_1, b_2, \cdots, b_m, Ab_1, \cdots, Ab_m, A^2b_1, \cdots\}$$

$n$ linearly independent column vectors are picked up s equentially and the set of these vectors be $S$.

$$S = [b_1, b_2, \cdots, b_m, Ab_1, \cdots] \tag{2.4}$$

and $\delta_i$ is given by

$$\delta_i = \max\{j \mid A^{j-1}b_i \in S\} \tag{2.5}$$

$$\delta = \max\{\delta_1, \delta_2, \cdots, \delta_m\} \tag{2.6}$$

The following theorem related to the disk property of matrix will be adopted to assign the poles of the transformed system.

**[*Theorem 2.1*]** [8] (Gershgorin's theorem)

Let $\lambda$ be an eigenvalue of an arbitrary matrix $A = (a_{jk}) \in R^{n \times n}$. Then for some integer $j$ $(1 \le j \le n)$ we have

$$|a_{jj} - \lambda| \le |a_{j1}| + |a_{j2}| + \cdots + |a_{j,j-1}| + + |a_{j,j+1}| + \cdots + |a_{jn}| \tag{2.7}$$

For each $(j = 1, \cdots, n)$ the inequality (2.7) determines a closed circular disk in the complex $\lambda$ plane whose center is at $a_{jj}$ and radius is given by the expression on the right-hand side of (2.7). ■

Theorem 2.1 states that each of the eigenvalues of $A$ lies in one of these $n$ disks.

**[*Lemma 2.2*]** [7] For a linear multivariable system $(A, B, C)$ a state feedback $F$ can be found such that the characteristic equation of the feedback system has arbitrary real coefficient if and only if $(A, B)$ is controllable. ■

## 3. MAIN RESULT

The main concept to be considered is as follows. The original system is transformed into Luenberger's second controllable canonical form and the feedback control law of the transformed system is chosen by adopting the matrix property based on the Gershgorin's theorem.

By using Gershgorin's theorem we can establish the closed-loop matrix $A_{cl}$ of the original system (2.1) such that all its poles are located in the unit disk. The feedback law $F$ is obtained from equation

$$F = B^{-1}(A_{cl} - A) \tag{3.1}$$

Unfortunately, the matrix $B$ not always has inverse, so we transform the original system into Luenberger's second controllable canonical form, in which the feedback law can be determined without solving the inverse problem. We have

$$\begin{aligned} \overline{A} + \overline{B}\,\overline{F} &= T^{-1}(A + BF)T \\ &= T^{-1}AT + T^{-1}BFT \end{aligned} \tag{3.2}$$

and $\overline{A} = T^{-1}AT;\ \overline{B} = T^{-1}B;\ \overline{F} = FT$

On the other hand, from [7] yields

$$\overline{A} + \overline{B}\,\overline{F} = \begin{bmatrix} E_1 \\ a_1^T + f_1^T \\ E_2 \\ \cdot \\ \cdot \\ \cdot \\ a_m^T + f_m^T \end{bmatrix} \tag{3.3}$$

where,

$$\overline{A} = [E_1^T, a_1, E_2^T, a_2, \cdots, E_m^T, a_m]^T \tag{3.4}$$

$$E_i = [\underbrace{0, \cdots, 0, I_{\delta_i - 1}}_{\sum_{j=1}^{i-1} \delta_j + 1}, 0, \cdots, 0] \} \ \delta_i - 1$$

$$a_i^T = [\alpha_{i0}, \alpha_{i1}, \cdots, \alpha_{i(n-1)}], \ (i = 1, \cdots, m)$$

$$\overline{B} = [0, b_1, 0, b_2, \cdots, 0, b_m]^T \tag{3.5}$$

$$b_i^T = [0, \cdots, 0, 1, \beta_{i(i+1)}, \cdots, \beta_{im}], \ (i = 1, \cdots, m)$$

and the state feedback is

$$\overline{F} = \overline{B}_m^{-1} \overline{F}_m \tag{3.6}$$

$$\overline{B}_m = \begin{vmatrix} b_1^T \\ \cdot \\ \cdot \\ b_m^T \end{vmatrix}, \quad \overline{F}_m = \begin{vmatrix} f_1^T \\ \cdot \\ \cdot \\ f_m^T \end{vmatrix},$$

where,

$$f_i^T = -a_i^T + [\underbrace{0, \cdots, 0, 1}_{\sum_{j=1}^{i} \delta_j}, 0, \cdots, 0] \tag{3.7}$$

$$f_m^T = -a_m^T + [-\gamma_0, -\gamma_1, \cdots, -\gamma_{n-1}]$$
$$(i = 1, \cdots, m-1) \tag{3.8}$$

In this canonical form, the feedback law $\overline{F}$ is determined by (3.7) and (3.8) without presence of $B$, but satifies

$$det\,[sI - (A + BF)] = s^n + \gamma_{n-1} s^{n-1} + \ldots + \gamma_0$$

If $\overline{A}, \overline{B}$ and $\overline{F}$ are determined by (3.4)-(3.6) respectively, so $(\overline{A} + \overline{B}\,\overline{F})$ has the form

$$\overline{A} + \overline{B}\,\overline{F} = \begin{bmatrix} 0 & 1 & . & . & . & 0 \\ \cdot & & & & & \cdot \\ \cdot & & & & & \cdot \\ \cdot & & & & & \cdot \\ 0 & & & & & 1 \\ -\gamma_0 & . & . & . & & -\gamma_{n-1} \end{bmatrix} \tag{3.9}$$

Let us discuss the closed loop matrix (3.9). From the theorem 2.1, it is known that the eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_{n-1}$ of the matrix $(\overline{A} + \overline{B}\,\overline{F})$ are located in the unit circle. The position of the last eigenvalue $\lambda_n$ depends on the values $\gamma_i$. If $\gamma_i$ $(i = 0, \cdots, n-1)$ can be chosen satisfying

$$\sum_{i=0}^{n-1} |\gamma_i| \le 1 \tag{3.10}$$

the eigenvalue $\lambda_n$ is located in the unit circle too. Therefore we can assign all the poles of the closed loop system $(\overline{A}, \overline{B})$ in the unit circle. From now we can get the pole assignment procedure. This procedure has the following steps

**Step 1** Transform the system $(A, B)$ to the Luenberger's second controllable canonical form $(\overline{A}, \overline{B})$.

**Step 2** Assign all the poles of the closed loop canonical system$(\overline{A}, \overline{B})$ in the unit circle.

**Step 3** Determine the feedback $F$ for the given system $(A, B)$.

## 4. NUMERICAL EXAMPLES

If the input matrix $B$ in the systems (2.1) is square and invertible so the pole assignment problem can be realized directly by using theorem 2.1. Consider the system

$$A = \begin{bmatrix} 1 & 0 & -3 \\ 3 & 1 & 0 \\ 0 & -2 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & -3 \\ -1 & 1 & 2 \end{bmatrix}$$

We want to place the poles in the unit circle. The closed loop matrix can be chosen, for example

$$A_{cl} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0.5 \\ 0.5 & 0.3 & 0 \end{bmatrix}$$

so from (3.1) yields

$$F = \begin{bmatrix} 0.3750 & -0.4750 & -0.2500 \\ -1.3750 & 1.4750 & 3.2500 \\ 1.1250 & 0.1750 & -0.2500 \end{bmatrix}$$

and $eig(A+BF) = [-0.3545 \pm 0.4764i, \ 0.7090]$

Now we consider the linear discrete system

$$A = \begin{bmatrix} 2.5 & 1 & 0 \\ 0 & 2.5 & 1 \\ -1 & -1 & 1.5 \end{bmatrix}, \quad B = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & -0.5 \\ 0.5 & 0.5 \end{bmatrix}$$

The Luenberger's second controllable canonical system

$$\overline{A} = \begin{bmatrix} 0 & 1 & 0 \\ -7.25 & 5 & 0 \\ 2 & 0 & 1.5 \end{bmatrix}, \quad \overline{B} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

If choosing $[\gamma_0, \gamma_1, \gamma_2] = [0.4, 0.4, 0.2]$, we have

$$F = \begin{bmatrix} -4.1250 & -6 & 0.1250 \\ -1.4000 & 1.2000 & 0.5000 \end{bmatrix}$$

and $eig(A+BF) = [-0.4 \pm 0.4899i, 1]$.

If choosing $[\gamma_0, \gamma_1, \gamma_2] = [0.1, 0.1, 0.8]$, $F$ becomes

$$F = \begin{bmatrix} -4.1250 & -6 & 0.1250 \\ -3.0750 & 2.4000 & 0.8750 \end{bmatrix},$$

and

$eig(A+BF) = [0.3428, \ -0.3850, \ -0.7578]$.

## 5. CONCLUSION

In this paper a method to assign all the poles of the closed loop system in the unit circle for linear discrete systems is presented. The base of the proposed method is the Luenberger's canonical form and the Gershgorin's theorem. The method is simple when a system has the same number of states and inputs.

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