Proceedings of the 12th
KACC, October 1997

LIMITING ZEROS OF SAMPLED SYSTEMS WITH APPROXIMATED FRACTIONAL ORDER HOLD

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Abstract. This paper is concerned with the properties of zeros of discrete-time systems which are composed of a hold, a continuous-time plant and a sampler in cascade. Here the signal reconstruction is based on the fractional order hold. In order to overcome the implementing problem of the fractional order hold, the piecewise constant reconstruction method by use of the zero order hold is introduced. The properties of the zeros are explored in the limiting cases when the sampling period tends to zero. The stability conditions of the zeros for sufficiently small sampling periods are also presented.

Keywords. linear systems, sampled systems, zeros, fractional order hold.

1. INTRODUCTION

In the design of control systems, the existence of unstable zeros makes it difficult to apply some control algorithms, such as model reference adaptive control and model matching. When a continuous-time system is discretized by use of a sampler and a hold, the stability of the zeros are not necessarily preserved. This problem was first studied by Keviczky and Kumar [1], Tuschák [2] and Åström *et al.* [3]. The zero locations of sampled systems were explored in the limiting cases when the sampling period tends to zero or infinity. These zeros are called the limiting zeros. The results have been extended by Hagiwara *et al.* [4]. In many of the above papers, the zero-order hold (ZOH) has been employed as a hold circuit since it is used most frequently in practice.

Passino *et al.* [5] have considered sampled systems with the fractional order hold (FROH) instead of the ZOH and shown that it can locate the zeros of the sampled systems inside the unit disc in some cases when the ZOH fails to do so. The stability conditions of the limiting zeros have been analyzed in the case of FROH by Ishitobi [6], [7]. However, it is not easy to design the FROH exactly in practice. One simple way of overcoming it is to reconstruct the FROH signal approximately by ZOH.

The purpose of this paper is to investigate the properties of the limiting zeros when the approximated FROH is used as a hold.

2. DISCRETE-TIME SYSTEMS WITH APPROXIMATED FRACTIONAL ORDER HOLD

Suppose that the state space equation of an nth-order time-invariant single-input single-output controllable and observable system is expressed as

$$\begin{cases} \dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) + \boldsymbol{b}u(t), \quad \boldsymbol{x}(t) \in \boldsymbol{R}^{n\times 1} \\ y(t) = \boldsymbol{c}^T\boldsymbol{x}(t) \end{cases} \tag{1}$$

where $u(t)$ and $y(t)$ are the input and the output scalars, respectively, and $\boldsymbol{x}(t)$ is the state vector.

We are interested in a discrete-time system composed of a hold circuit, the above continuous-time system and a sampler in series. When the fractional order hold (FROH) signal reconstruction method is considered, the input is described by

$$u(t) = u(kT) + \beta\left[\frac{u(kT) - u((k-1)T)}{T}\right](t - kT), \quad kT \le t < (k+1)T \tag{2}$$

where T is the sampling period and the parameter β is a real number [5], [8]. The fractional order holds with $\beta = 0$ and with $\beta = 1$ are identical to the zero order hold and the first order hold (FOH), respectively.

The transfer function of FROH is expressed as

$$G_\beta(s) = \left(1 - \beta e^{-Ts}\right)\frac{1-e^{-Ts}}{s} + \frac{\beta}{Ts^2}\left(1-e^{-Ts}\right)^2 \quad (3)$$

It has been shown that FROH is superior to ZOH in some cases with respect to the stability of zeros of sampled systems [5]-[7]. However, it is not easy to implement the ideal FROH in practice. We propose here approximation method of FROH by use of a series of piecewise constant signal reconstruction as shown in Fig.1.

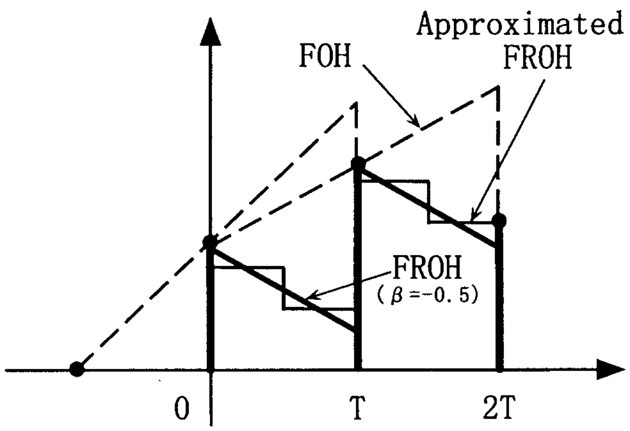

Fig. 1. Signal reconstruction of approximated FROH with $N = 2$ and $\beta = -0.5$.

When the approximated FROH is used, the input is represented as

$$\begin{aligned} u(t) &= \left(1 + \frac{2\ell - 1}{2N}\beta\right)u(kT) \\ &\quad - \frac{2\ell-1}{2N}\beta u((k-1)T) \\ &\quad \left(k + \frac{\ell - 1}{N}\right)T \le t < \left(k + \frac{\ell}{N}\right)T \\ &\quad \ell = 1, \cdots, N \end{aligned} \quad (4)$$

The transfer function of the approximated FROH is given by

$$G_{N\beta}(s) = \frac{1}{2N}\left\{\frac{\beta(1-e^{-sT})(1+e^{-sT/N})}{1-e^{-sT/N}} + 2N(1-\beta e^{-sT})\right\}\frac{1-e^{-sT}}{s} \quad (5)$$

Hence, the sampled transfer function $H_{N\beta}(z)$ with approximated FROH becomes

$$\begin{aligned} H_{N\beta}(z) &= \mathcal{Z}\left[G_{N\beta}(s)G(s)\right] \\ &= \frac{2N+\beta}{2N}\mathcal{Z}\left[\frac{1-e^{-Ts}}{s}G(s)\right] \\ &\quad + \frac{\beta}{N}\mathcal{Z}\left[\frac{1-e^{-Ts}}{s}\sum_{i=1}^{N-1} e^{-iTs/N}G(s)\right] \\ &\quad - \frac{(2N-1)\beta}{2N}\mathcal{Z}\left[\frac{1-e^{-Ts}}{s}e^{-Ts}G(s)\right] \end{aligned} \quad (6)$$

where $G(s) = \boldsymbol{c}^T(sI - A)^{-1}\boldsymbol{b}$ is the transfer function of a plant.

When the piecewise constant signal reconstruction with two stages is applied for approximation of FROH, namely $N = 2$, the state space representation of a sampled system is obtained as follows.

$$\left\{\begin{aligned} \begin{bmatrix} \boldsymbol{x}((k+1)T) \\ x_1((k+1)T) \end{bmatrix} &= \begin{bmatrix} \Phi & \boldsymbol{\phi} \\ \boldsymbol{0}^T & 0 \end{bmatrix}\begin{bmatrix} \boldsymbol{x}(kT) \\ x_1(kT) \end{bmatrix} \\ &\quad + \begin{bmatrix} \boldsymbol{\psi} \\ 1 \end{bmatrix} u(kT) \\ y(kT) &= [\boldsymbol{c}^T \;\; 0]\begin{bmatrix} \boldsymbol{x}(kT) \\ x_1(kT) \end{bmatrix} \end{aligned}\right. \quad (7)$$

where

$$\Phi = e^{AT} = \sum_{i=0}^{\infty}\frac{(AT)^i}{i!} \quad (8)$$

$$\boldsymbol{\phi} = -\frac{\beta}{4}(\boldsymbol{\gamma} + 3\boldsymbol{\lambda}) \quad (9)$$

$$\boldsymbol{\psi} = \left(1 + \frac{\beta}{4}\right)\boldsymbol{\gamma} + \left(1 + \frac{3\beta}{4}\right)\boldsymbol{\lambda} \quad (10)$$

$$\begin{aligned} \boldsymbol{\gamma} &= \int_0^{T/2} e^{A(T-\tau)}\boldsymbol{b}d\tau \\ &= \sum_{i=0}^{\infty}\left(1 - \frac{1}{2^{i+1}}\right)\frac{(AT)^i}{(i+1)!}\boldsymbol{b}T \\ &= \boldsymbol{\mu} - \boldsymbol{\lambda} \end{aligned} \quad (11)$$

$$\begin{aligned} \boldsymbol{\lambda} &= \int_{T/2}^{T} e^{A(T-\tau)}\boldsymbol{b}d\tau \\ &= \sum_{i=0}^{\infty}\frac{(AT)^i}{2^{i+1}(i+1)!}\boldsymbol{b}T \end{aligned} \quad (12)$$

$$\boldsymbol{\mu} = \sum_{i=0}^{\infty}\frac{(AT)^i}{(i+1)!}\boldsymbol{b}T \quad (13)$$

and $\boldsymbol{0}$ is an nth column vector with all zero elements.

Notice here the expression of infinite series expansions (8)-(13), then we get

$$\begin{bmatrix} \Phi & \boldsymbol{\mu} \\ \boldsymbol{0}^T & 1 \end{bmatrix} = \exp\begin{bmatrix} AT & \boldsymbol{b}T \\ \boldsymbol{0}^T & 0 \end{bmatrix} \quad (14)$$

$$\begin{bmatrix} * & \boldsymbol{\lambda} \\ \boldsymbol{0}^T & 1 \end{bmatrix} = \exp\begin{bmatrix} \frac{AT}{2} & \frac{\boldsymbol{b}T}{2} \\ \boldsymbol{0}^T & 0 \end{bmatrix} \quad (15)$$

The above relations imply that the model can be converted from continuous-time to discrete-time using a matrix exponential. A MATLAB version of the algorithm is given in Appendix.

3. PROPERTIES OF LIMITING ZEROS

The zeros of a pulse transfer function are often called the limiting zeros when the sampling period goes to zero or infinity [3]. In this section, the limiting zeros of $H_{N\beta}(z)$ are characterized. We have the following theorem about the zeros in the limiting case when a sampling period tends to zero.

Theorem 1 : Suppose that $G(s)$ is a strictly proper nth order transfer function expressed as

$$G(s) = \frac{K(s+r_1)\cdots(s+r_m)}{(s+a_1)\cdots(s+a_n)}, \quad K \neq 0 \tag{16}$$

Then, $H_{N\beta}(z)$ approaches

$$\frac{KT^{n-m}}{(n-m)!}\frac{(z-1)^m E_{n-m}(z;\ \beta)}{z(z-1)^n} \tag{17}$$

as $T \to 0$, where

$$\begin{aligned} E_p(z;\ \beta) &= \frac{2N+\beta}{2N}B_p(z) + \frac{\beta}{N}\sum_{i=1}^{N-1} B_p\left(z, \frac{i}{N}\right) \\ &\quad -\frac{(2N-1)\beta}{2N}B_p(z,1) \end{aligned} \tag{18}$$

$$B_p(z) = b_{p1}z^{p-1} + b_{p2}z^{p-2} + \cdots + b_{pp} \tag{19}$$

$$b_{pk} = \sum_{i=1}^{k}(-1)^{k-i}i^p\begin{pmatrix} p+1 \\ k-i \end{pmatrix}, \quad k = 1, \ldots, p \tag{20}$$

$$B_p(z,\Delta) = b_{p0}(\Delta)z^p + b_{p1}(\Delta)z^{p-1} + \cdots + b_{pp}(\Delta) \tag{21}$$

$$\begin{aligned} b_{pk}(\Delta) &= \sum_{i=0}^{p-k}(-1)^{p-k-i}(i+\Delta)^p\begin{pmatrix} p+1 \\ p-k-i \end{pmatrix}, \\ &\quad k = 0, \ldots, p \end{aligned} \tag{22}$$

and a better resolution of the m zeros close to 1 is given by $\exp(-r_iT)$.

Remark 1 : The polynomials $E_p(z;\ \beta)$ are listed below for a few cases.

Case (i) $p = 1$:

$$E_1(z;\ \beta) = \frac{2+\beta}{2}z - \frac{\beta}{2} \tag{23}$$

Case (ii) $p = 2$:

$$\begin{aligned} E_2(z;\ \beta) &= \left(1 + \frac{2N^2+1}{6N^2}\beta\right)z^2 \\ &\quad + \left(1 + \frac{N^2-1}{3N^2}\beta\right)z - \frac{4N^2-1}{6N^2}\beta \end{aligned} \tag{24}$$

Case (iii) $p \geq 3$:
When $N = 2$

$$\begin{aligned} E_p(z;\ \beta) &= \frac{4+\beta}{4}zB_p(z) + \frac{\beta}{2}B_p\left(z, \frac{1}{2}\right) \\ &\quad -\frac{3\beta}{4}B_p(z,1) \\ &= \left\{\frac{4+\beta}{4} + \frac{\beta}{2}\left(\frac{1}{2}\right)^p\right\}z^p \\ &\quad + \left[\left\{\frac{4+\beta}{4} + \frac{\beta}{2}\left(\frac{1}{2}\right)^p\right\}(2^p - p - 1)\right. \\ &\quad \left. +\frac{\beta}{2}\left\{\left(\frac{3}{2}\right)^p - \frac{5}{2}\right\}\right]z^{p-1} \\ &\quad + \cdots + \frac{\beta}{2}\left(\frac{1}{2}\right)^p - \frac{3\beta}{4} \end{aligned} \tag{25}$$

4. STABILITY OF ZEROS

In this section, the stability of the zeros is studied on the basis of the results of the previous section. The stability condition of the zeros for a sufficiently small T is shown by the following theorem.

Theorem 2 : Suppose that $G(s)$ has no zeros on the imaginary axis. Let p denote the relative degree of $G(s)$.

Case (i) $p = 1$:

- All the zeros of $H_{N\beta}(z)$ for a sufficiently small T are stable only if all the zeros of $G(s)$ are stable and

$$-1 \leq \beta \tag{26}$$

- If all the zeros of $G(s)$ are stable and

$$-1 < \beta \tag{27}$$

then all the zeros of $H_{N\beta}(z)$ for a sufficiently small T are stable.

Case (ii) $p = 2$:

- All the zeros of $H_{N\beta}(z)$ for a sufficiently small T are stable only if all the zeros of $G(s)$ are stable and

$$-1 \leq \beta \leq 0 \tag{28}$$

- If all the zeros of $G(s)$ are stable and

$$-1 < \beta < 0 \tag{29}$$

then all the zeros of $H_{N\beta}(z)$ for a sufficiently small

T are stable.

Case (iii) $p \geq 3$:

- When $N = 2$, at least one of the zeros of $H_{N\beta}(z)$ for a sufficiently small T is unstable.

Remark 2 : If the relative degree of a continuous-time transfer function is two and the sum of the zeros is less than or equal to the sum of the poles, then the zeros of the sampled system with the approximated FROH of $-1 < \beta < 0$ stay definitely inside the unit disc for sufficiently small sampling periods while at least one zero of those with ZOH lies outside or on the unit disc. This result is the same as that of the ideal FROH.

5. AN EXAMPLE

Consider a continuous-time transfer function

$$G(s) = \frac{s+1}{s^3} \tag{30}$$

The stability of zeros of the corresponding sampled system with three types of holds is given as follows.

Case (i) ZOH:

- One of two zeros is located outside the unit disc for all sampling periods; i.e., unstable.

Case (ii) Ideal FROH:

- All the three zeros stay inside the unit disc for $T \leq 2.0$.

Case (iii) Approximated FROH with $N = 2$ and $\beta = -0.5$:

- All the three zeros lie inside the unit disc for $T \leq 1.5$.

The zeros are stable for small sampling periods in the cases of (ii) and (iii) though the sampling range for stable zeros of the case (iii) is smaller than that of the case (ii).

6. CONCLUSIONS

A sampled system with the approximated fractional order hold is treated. The fractional order hold is implemented by use of the zero order hold on the basis of the piecewise constant signal reconstruction method. The stability of the limiting zeros of sampled systems with the approximated fractional order hold is improved compared with the zero order hold. The result is the same as that of the ideal fractional order hold.

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7. APPENDIX

```
function [Phi, Gamma, Psi, Eta] = ...
                 c2db2st(a, b, c, d, t, beta)
%[Phi,Gamma,Psi,Eta]=C2DB2ST(A,B,C,D,T,Beta)
%converts the continuous-time system:
%
%       dx/dt = Ax + Bu
%       y     = Cx + Du
%
%to the discrete-time state-space system:
%
%       z[n+1] = Phi * z[n] + Gamma * u[n]
%       y[n]   = Psi * z[n] + Eta * u[n]
%
%assuming a fractional-order hold
%approximated by 2 staircases with a
%parameter beta on the inputs and sample
%time T.
error(nargchk(6,6,nargin));
error(abcdchk(a,b,c,d));
[m,n] = size(a);
[m,nb] = size(b);
[nc,n] = size(c);
s1 = expm([[a b]*t; zeros(nb,n+nb)]);
s2 = expm([[a/2 b/2]*t; zeros(nb,n+nb)]);
Phib = -beta*(s1(1:n,n+1:n+nb)+...
              2*s2(1:n,n+1:n+nb))/4;
Phi = [s1(1:n,1:n) Phib; zeros(nb,n+nb)];
Gamma = [(1+beta/4)*s1(1:n,n+1:n+nb)+...
          beta*s2(1:n,n+1:n+nb)/2;
         eye(nb)];
Psi = [c zeros(nc,nb)];
Eta = d;
```