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The Pulse Response Circulant Matrix: Application to the Design of Reduced-Order Robust MPC and MIMO Identification

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Abstracts Two different issues, design of reduced-order robust model predictive control and input signal design for identification of a MIMO system, are addressed and design techniques based on singular value decomposition(SVD) of the pulse repsonse circulant matrix(PRCM) are proposed. For this, we investigate the properties of the PRCM, which is a periodic approximation of a linear discrete-time system, and show its SVD represents the directional as well as the frequency decomposition of the system. Usefulness of the PRCM and effectiveness of the proposed design techniques are demonstrated through numerical examples.

Keywords Pulse repsonse circulant matrix, Circulant matrix, Model predictive control, Identification

1. Introduction

Traditionally, the frequency domain analysis has provided us with deeper insight into the behaviors of linear systems and enabled us to come up with many useful techniques for control and identification. However, the frequency domain methods are sometimes revealed to be cumbersome or hard to realize, especially for MIMO systems. To extend the advantages of the frequency domain methods to MIMO systems as well, there needs to be a bridge which can conveniently link the theoretical results obtained in the frequency domain to practical time domain algorithms. The objective of this research lies in pursuing this issue by using the so-called pulse response circulant matrix(PRCM). To demonstrate the usefulness of the PRCM, we consider the following two problems as motivating issues: design of reduced-order robust model predictive control(MPC) and design of input signal for MIMO identification. Before introducing what are meant by these problems, we define the underlying MIMO system description.

Consider an n_u-input/n_y-output MIMO time-invariant discrete-time system.

$$\mathbf{y}(z^{-1}) = \mathbf{H}(z^{-1})\mathbf{u}(z^{-1}) \quad (1)$$

Frequency response of the system can be obtained by simply substituting z with $e^{j\omega}$.

$$\mathbf{y}(e^{-j\omega}) = \mathbf{H}(e^{-j\omega})\mathbf{u}(e^{-j\omega}), \quad \omega \in [0, 2\pi) \quad (2)$$

$\mathbf{H}$ is symmetric with respect to the Nyquist frequency, $\omega = \pi$. Let the singular value decomposition(SVD) of $\mathbf{H}$ be $\mathbf{H} = \mathbf{WDV}^T$ where each matrix is a function of $e^{-j\omega}$. Here, the diagonal elements of $\mathbf{D}$ represent the directional gain of the system at each frequency.

Design of Robust Reduced-Order MPC: In MPC, input moves over a control horizon are calculated at each time instance such that the predicted output error over a prediction horizon is minimized. This step usually requires heavy computational load, especially as the number of inputs as well as the length of the control horizon increase. Nevertheless, when there is model uncertainty, the output prediction goes awry and the resulting input moves may lose its meaning. When the model uncertainty arises in a high-frequency range, which is the usual case, insertion of a low-pass filter in the feedback path can enhance the robustness at the expense of performance. However, this method not only imposes an additional phase lag on the feedback loop but also asks additional computation. One of the techniques to implement this idea while not sufferring these drawbacks is to approximate the future input moves as a linear combination of low-frequency sinusoids and solve the QP(quadratic programming) with respect to the associated coefficients. In fact, a similar technique has already been studied under the name of blocking[2] with wavelet basis instead of sinusoids. The study, however, has been carried out for SISO systems in the context of computation reduction.

In MIMO cases, directionality is one more aspect that needs to be considered relating to this issue. The low-gain directions are very often not easy to identify. Moreover, in most regulation problems, the outputs need not be controlled along low-gain directions, particularly at high frequencies. Brute attempt to control all the modes of a system may lead to poor closed-loop performance and sometimes even to instability[?]. Since complete information on the directional gain is contained in the SVD of $\mathbf{H}(e^{-j\omega})$, we can identify the input directions from $\mathbf{V}(e^{-j\omega})$ corresponding to desired directional gains. Hence, it can be thought that the input moves composed of a linear combination of the desired directions will solve this problem with additional opportunity to further reduce the dimensionality of the QP. Conceptually, the idea sounds obvious but the detailed procedure how to implement is not clear, though.

Design of Inputs for MIMO Identification: One of the recent issues in control-relavent identification has been arisen concerning the directionality of a MIMO system. For appropriate design of a MIMO control system, correct identification of the directional gains is more important than the element-wise accurate identification of the transfer function matrix. For proper identification of low directional gains, it is necessary to sufficiently excite the low directional modes of the system. So far, two different approaches have been studied concerning this issue; input design from open-loop experiments[4] and automatic input generation through closed-loop experiments[3]. Currently, the former method has been limited only to resolving the steady state directionality. Extension to dynamic state identification has been performed only heuristically. On the other hand, the latter approach can resolve directionality up to dynamic modes. However, extension to MIMO systems with more than two inputs and two outputs is rather cumbersome and the requirement of tightly-tuned PI control is thought to be a bit stringent. In this paper, we are concerned with the first approach but with generalization to the identification of dynamic states. Basically, the underlying idea is quite simple.

The system in (2) under random output noise can be written as

$$\mathbf{y}(e^{-j\omega}) = \mathbf{H}(e^{-j\omega})\mathbf{u}(e^{-j\omega}) + \mathbf{E}(e^{-j\omega}) \tag{3}$$

Substituting the SVD into this equation leads to

$$(\mathbf{W}^T\mathbf{y}) = \mathbf{D}(\mathbf{V}^T\mathbf{u}) + (\mathbf{W}^T\mathbf{E}) \tag{4}$$

Let $\mathbf{w}_i$ and $\mathbf{v}_i$ be the i^{th} column vectors of $\mathbf{W}$ and $\mathbf{V}$, and d_i be the i^{th} diagonal element of $\mathbf{D}$, respectively. Assume that the noise projected on each output principal direction has the same magnitude. Under this condition, we can see that the input

$$\mathbf{u}(e^{-j\omega}) = \alpha(\omega)\sum_i(\mathbf{v}_i(e^{-j\omega})/d_i(e^{-j\omega})) \tag{5}$$

yields the same signal to noise ratio to all output principal directions at frequency ω. Linearly combining the inputs obtained at different frequencies and then converting it into the time domain will produce an input signal which gives unbiased excitation in all output directions. However, this approach in itself is not adequate to directly use unless a more systematic and numerically efficient procedure is devised.

Particularly motivated by the above two issues, the objective of this research has been placed in the development of a consistent and numerically efficient way which can bridge design concepts for control and identification in the frequency domain and their time domain realization. It is believed that the so-called pulse response circulant matrix (PRCM) can play this role. In the following sections, we first introduce what the PRCM is and then investigate some important properties of the PRCM focusing on the relationship between the SVD of the PRCM and the frequency response of the system.

2. Pulse Response Circulant Matrix

2.1. Definitions and Some Fundamental Properties

Consider the system given in (1). Let $\mathbf{h}_k \in R^{n_y \times n_u}$, $k = 1, 2\cdots$ be the k^{th} pulse response of the system. Now consider the periodic approximation of the system where an input sequence with the period N produces a periodic output with the same period. It is easy to see that such a system can be represented by the following matrix form:

$$\begin{bmatrix} \mathbf{y}(k+1) \\ \mathbf{y}(k+2) \\ \vdots \\ \mathbf{y}(k+N) \end{bmatrix} = \begin{bmatrix} \mathbf{h}_1 & \mathbf{h}_N & \cdots & \mathbf{h}_2 \\ \mathbf{h}_2 & \mathbf{h}_1 & \cdots & \mathbf{h}_3 \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{h}_N & \mathbf{h}_{N-1} & \cdots & \mathbf{h}_1 \end{bmatrix} \begin{bmatrix} \mathbf{u}(k) \\ \mathbf{u}(k+1) \\ \vdots \\ \mathbf{u}(k+N-1) \end{bmatrix}$$

$$\rightarrow \quad \mathbf{Y}_N = \mathbf{H}_N\mathbf{U}_N \tag{6}$$

It is obvious that (6) $\rightarrow$ (1) as $N \rightarrow \infty$. We call $\mathbf{H}_N$ the pulse response circulant matrix(PRCM).

Discrete Fourier transform(DFT) of (6) converts the equation into the frequency domain. For N discrete $m-$dimensional vector sequences, the DFT operator can be described by the following $mN \times mN$ matrix:

$$\begin{aligned} \mathcal{F}_m &= [f_{k,l}] \text{ where } f_{k,l} = (1/\sqrt{N})e^{-j\omega_{k-1}(l-1)}\mathbf{I}_m, \\ & \omega_k = (2\pi/N)k, \quad k,l = 1,2,\cdots N \end{aligned} \tag{7}$$

where $\mathbf{I}_m$ is the $m \times m$ identity matrix. Taking the DFT on both sides of (6) yields

$$(\mathcal{F}_{n_y}\mathbf{Y}_N) = (\mathcal{F}_{n_y}\mathbf{H}_N\bar{\mathcal{F}}_{n_u})\ (\mathcal{F}_{n_u}\mathbf{U}_N) \ \rightarrow\ \mathcal{Y}_N = \mathcal{H}_N\mathcal{U}_N \tag{8}$$

where $^-$ denotes the complex conjugate. The PRCM and its Fourier transform have the following fundamental properties:

1. For SISO systems, diagonal elements of $\mathcal{H}_N$ are the eigenvalues of $\mathbf{H}_N$[1].
2. Using (8) and extending the above property, it can be shown that

$$\mathcal{H}_N = diag[\hat{\mathbf{h}}_k],\ \hat{\mathbf{h}}_k = \sum_{l=1}^{N}\mathbf{h}_l e^{-j\omega_k l} \in C^{n_y \times n_u} \tag{9}$$

For large N, $\mathbf{H}(e^{-j\omega}) \approx \sum_{l=1}^{N}\mathbf{h}_l e^{-j\omega l}$. Hence, we know that $\mathcal{H}_N$ contains $\mathbf{H}(e^{-j\omega})$ evaluated at ω_k as its diagonal element. As $N \rightarrow \infty$, $\mathcal{H}_N \rightarrow diag\left[\mathbf{H}(e^{-j\omega})\right]$.

2.2. Properties of SVD of PRCM

SISO case : First we consider the case where $n_u = n_y = 1$. Let's represent the SVD of $\mathbf{H}_N$ as

$$\mathbf{H}_N = \mathbf{W}_N\mathbf{D}_N\mathbf{V}_N^T \tag{10}$$

From (8),

$$\mathcal{H}_N = (\mathcal{F}\mathbf{W}_N)\mathbf{D}_N(\bar{\mathcal{F}}\mathbf{V}_N) \tag{11}$$

where we drop the subscript $_1$ from $\mathcal{F}$ for notational simplicity. Since both $\mathcal{F}\mathbf{W}_N$ and $\bar{\mathcal{F}}\mathbf{V}_N$ are unitary, it is evident that the above is the SVD of $\mathcal{H}_N$. Moreover, since $\mathcal{H}_N$ is diaginal, we know that $\mathbf{D}_N$ is the absolute value (elementwise) of $\mathcal{H}_N$ with its elements rearranged in the order of descending magnitude. In other words, $\mathbf{D}_N$ represents the amplitude ratio of the original system for a large N. Also, from the property of the DFT matrix, it can be shown that $\hat{\mathbf{h}}_k = \hat{\mathbf{h}}_{N-k}$, $k = 1, 2, \cdots, N-1$. Hence, $\mathbf{D}_N$ has the same element at two consecutive positions except for $k = 0$. Now let $\mathbf{v}_m$ and $\mathbf{v}_{m+1}$ be the input singular vectors corresponding to two identical singular values. Also, let the associated frequency with the singular value be ω_k. Then through somewhat tedious but straightforward manipulations, we can show that

$$\mathbf{v}_m = \frac{1}{\sqrt{N}}\begin{bmatrix} \cos(0) \\ \cos(\omega_k) \\ \vdots \\ \cos(\omega_k(N-1)) \end{bmatrix} \tag{12}$$

and $\mathbf{v}_{m+1}$ is the sine complement of $\mathbf{v}_m$. The indices m and $m+1$ may be interchanged. Similar results can be obtained for $\mathbf{w}_m$ and $\mathbf{w}_{m+1}$. It can be shown that

$$\mathbf{w}_m = \frac{1}{\sqrt{N}}\begin{bmatrix} \cos(-\phi_k) \\ \cos(\omega_k - \phi_k) \\ \vdots \\ \cos(\omega_k(N-1) - \phi_k) \end{bmatrix} \tag{13}$$

and $\mathbf{w}_{m+1}$ is the sine complement of $\mathbf{w}_m$ where ϕ_k is the phase angle by the system at ω_k, *i.e.*, $h(e^{-j\omega_k}) = r_k e^{-j\phi_k}$.

Summerizing the above, we know that for a SISO system the SVD of the PRCM when N is sufficiently large has the following properities:

1. The singular values represents the amplitude ratios of the system at discrete frequencies, $\omega_k = 2\pi k/N$.
2. Each of the right singular vectors is a discretized cosine or sine function of the frequency of the corresponding singular value. Same is true for the left singular vectors but with a phase lag at the corresponding frequency.

The above tells us that the SVD of the PRCM of a SISO system provides the complete information (but represented with real values) about the frequency response of the system.

MIMO Case : The analysis in the SISO case can be easily extended to the MIMO case. To state the conclusion first, SVD of the PRCM decomposes a MIMO system not only into the frequencies but also into the principal directions. To show this, we revisit (9) and let the SVD of $\hat{\mathbf{h}}_k(e^{-j\omega_k})$ be

$$\hat{\mathbf{h}}_k = \hat{\mathbf{w}}_k \hat{\mathbf{d}}_k \hat{\mathbf{v}}_k^T, \qquad (14)$$

Then we can rewrite (9) as

$$\mathcal{H}_N = diag[\hat{\mathbf{w}}_k] diag[\hat{\mathbf{d}}_k] diag[\hat{\mathbf{v}}_k^T] = \hat{\mathbf{W}}_N \hat{\mathbf{D}}_N \hat{\mathbf{V}}_N^T \qquad (15)$$

From the fact that $\hat{\mathbf{W}}_N$ and $\hat{\mathbf{V}}_N$ are unitary, and $\hat{\mathbf{D}}_N$ is real diagonal, we can see that (15) represents the SVD of $\mathcal{H}_N$ but not yet rearranged according to the magnitude. The elements of $\hat{\mathbf{D}}_N$ shows the directional gain (amplitude ratio) of the system at each frequency. Now, let the SVD of the PRCM be

$$\mathbf{H}_N = \mathbf{W}_N \mathbf{D}_N \mathbf{V}_N^T \qquad (16)$$

Substitution of (9) and (15) into (16) and rearrangement yields

$$(\mathcal{F}_{n_y} \hat{\mathbf{W}}_N) \hat{\mathbf{D}}_N (\mathcal{F}_{n_u} \hat{\mathbf{V}}_N)^T = \mathbf{W}_N \mathbf{D}_N \mathbf{V}_N^T \qquad (17)$$

Since both $\hat{\mathbf{D}}_N$ and $\mathbf{D}_N$ are real and diagonal, and the other four matrices, $\mathcal{F}_{n_y}\hat{\mathbf{W}}_N$, $\mathcal{F}_{n_u}\hat{\mathbf{V}}_N$, $\mathbf{W}_N$, and $\mathbf{V}_N$, are unitary, we can see that $\mathbf{D}_N$ is just a reordering of the diagonal elements of $\hat{\mathbf{D}}_N$ in the descending magnitude. Hence, $\mathbf{D}_N$ retains the property of $\hat{\mathbf{D}}_N$ as it is, and $\mathbf{V}_N$ and $\mathbf{W}_N$ contains the corresponding input and output principal directions at the corresponding frquency.

3. Applications

3.1. Design of Robust Reduced-Order MPC

In section 1, we have briefly discussed how the design of robust reduced-order MPC can be approached in the frequency domain. However, the idea was not easy to be practiced as it stands. In this subsection, we show the PRCM plays as a convenient vehicle in realizing the design concept. For this purpose, we assume that the control horizon, M, is chosen to be sufficiently long (at least up to the settling time of the process) such that the PRCM describes the process accurately enough. In fact, this choice agrees with the generally accepted MPC tuning rule.

We let $\Delta\mathbf{U}_{k|k}$ denote the future input moves that should be optimized by QP at k. The number of decision variables determined by QP is $n_u \times M$, which easily exceeds 100 in multi-input cases. Let $\mathbf{H}_M$ be the PRCM of the concerned process, which is composed of $\mathbf{h}_1$ through $\mathbf{h}_M$, and the SVD of $\mathbf{H}_M$ be

$$\mathbf{H}_M = \mathbf{W}_M \mathbf{D}_M \mathbf{V}_M \qquad (18)$$

Assume that the SVD is partitioned as

$$[\mathbf{W}_I \ \mathbf{W}_{II}] \begin{bmatrix} \mathbf{D}_I & 0 \\ 0 & \mathbf{D}_{II} \end{bmatrix} [\mathbf{V}_I \ \mathbf{V}_{II}]^T, \ \mathbf{D}_I \gg \mathbf{D}_{II} \qquad (19)$$

Hence,

$$\mathbf{Y}_M \approx \mathbf{W}_I \mathbf{D}_I \mathbf{V}_I^T \mathbf{U}_M \qquad (20)$$

This implies that the future input movements projected onto a lower-dimensional subspace, $\mathbf{V}_I^T\mathbf{U}_M$, virtually determines the future output movements. Since the column vectors of $\mathbf{V}_I$ are mutually orthogonal, we can approximate

$$\Delta\mathbf{U}_{k|k} \approx \mathbf{V}_I \mathbf{a}_{k|k} \equiv \mathbf{B}\mathbf{a}_{k|k} \qquad (21)$$

Here, $\mathbf{B}$ is called a blocking matrix[2]. Substituting the above into the MPC equations, the optimization problem is recast to have a lower order decision variable, *i.e.*, $\mathbf{a}_{k|k}$.

The robust design can be performed rather easily once the undesired frequency range is identified. Since the column vectors in $\mathbf{V}_I$ are discretized sine or cosine functions, we can accomplish the intended low-pass filtering by taking out column vectors from $\mathbf{V}_I$ associated with the unwanted frequency range. This approach has advantages over the conventional one using a low-pass filter in the feedback path in that no phase lag is added and required computation is further reduced as a bonus.

The following examples demonstrates how the proposed design technique performs:

Example 1: The process and the nominal model are zero-order hold equivalents of the following transfer functions with sampling interval of 2:

$$\mathbf{G}^{pro}(s) = \begin{bmatrix} \frac{1.7}{(10s+1)(10s^2+s+1)} & \frac{2.3}{30s+1} \\ \frac{1.3}{20s+1} & \frac{2.8}{(10s+1)(5s^2+s+1)} \end{bmatrix} \qquad (22)$$

$$\mathbf{G}^{nom}(s) = \begin{bmatrix} \frac{1.5}{10s+1} & \frac{2.2}{30s+1} \\ \frac{1.2}{20s+1} & \frac{2.6}{10s+1} \end{bmatrix} \qquad (23)$$

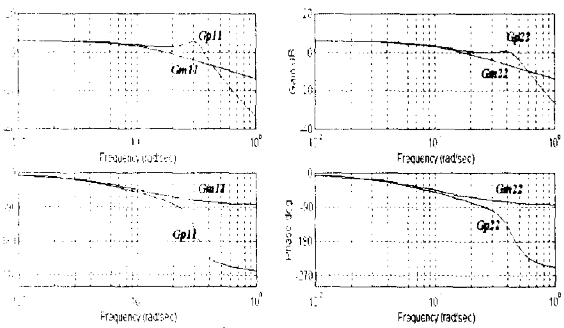
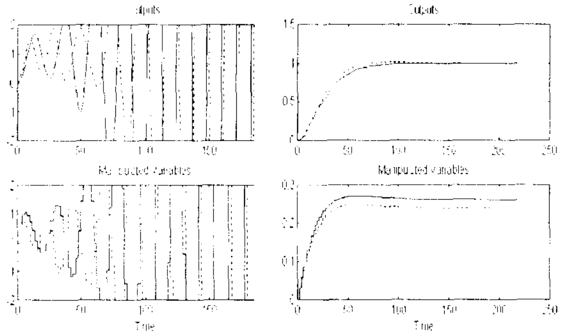


Fig. 1. Comparison of the frequency characteristics.

The prediction as well as control horizons were chosen to be 90, and both input and output weighting matries were chosen to be **I**. In Fig. 1, Bode plots for the diagonal elements of the process and the nominal model are compared. As can be seen, there exists significant modelling error beyond 0.1 rad/sec. In Fig.2, we compare the response of regular MPC and that of reduced-order MPC. For reduced-order design, we construct-ed a 180×180 PRCM using 90 2×2 pulse response coefficient matrices of the nominal model, and formulated the blocking matrix using the first 12 column vectors of the input singular matrix of the PRCM. Figure 2 compares the responses to set point change. While the regular MPC fails to converge due to significant model error, the reduced-order MPC recovers stability at some expense of dynamic vitality.



Regular MPC Reduced-order MPC

Fig. 2. Output responses to set point change.

3.2. Input Design for MIMO System Identification

In section 1, we have discussed how the input has to be designed in the frequency domain in order to uniformly excite the all the principal modes of a system. But, how to realize the signal as a time sequence was not clear. Here, we show that the PRCM enables us to directly substantiate the input sequence retaining the frequency domain requirements intact.

Consider the following relationship where the output is corrupted by random noise:

$$\mathbf{Y}_N = \mathbf{H}_N \mathbf{U}_N + \mathbf{E}_N = \mathbf{W}_N \mathbf{D}_N \mathbf{V}_N^T \mathbf{U}_N + \mathbf{E}_N \qquad (24)$$

To have proper estimation of all the principal gains, the input should be designed so that the signal to noise ratio of the resulting output is sufficiently large in every output principal direction. This can be achieved using the following input sequence:

$$\mathbf{U}_N \approx \sum_{k=1}^{n_u \times N} \alpha_k \frac{\mathbf{v}_k}{d_k} \gg \mathbf{W}_N^T \mathbf{E}_N \qquad (25)$$

for some large α_k. In practice, *a priori* information on $\mathbf{H}_N$ is assumed to be unavaliable. Under this situation, we may take the following steps:

step 1 Apply uniformly distributed excitations (*e.g.*, PRBS) to all inputs.

step 2 Identify the system and formulate an estimate of the PRCM, $\hat{\mathbf{H}}_N$. Take the SVD of $\hat{\mathbf{H}}_N$.

step 3 Construct the input signal according to (25) and apply it to the system.

step 4 Repeat step 2, 3 until $\hat{d}_k$s converge.

In the first run, only the modes associated with large singular values will be estimated correctly. Since the input directions for the incorrectly estimated low-gain modes are orthogonal to those of correctly estimated modes, repeated applications of (25) will lead to balanced excitation for every mode. We can concentrate the input energy on a certain desired frequency band, if exists, by adjusting α_k.

Example 2: We consider the zero-order hold equivalent of (22) with sampling period of 2 as the process to identify. It is assumed that zero-mean Gaussian random noise with variance of 0.5 is imposed on each outputs. Estimates of the process model

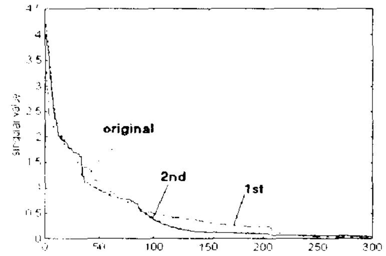


Fig. 3. Singular values of the process and its estimates.

were obtained using the subspace identification technique and PRCMs were derived from the estimated models with $N = 150$. In the first experiment, independent PRBS with amplitude 1.5 was applied for 500 sampling instances to each input. For the subsequent experiments, α_k was chosen to be 2. In Fig. 2, we compare the singular values of the true process PRCM with those of the estimates from the first and second experiments. We can see the singular value estimates converge to the true values almost perfectly in just two experiments.

4. Conclusions

In this paper, we have shown that the PRCM is a useful and convenient vehicle in translating various design concepts in the frequency domain for control and identification of a MIMO system into implementable algorithms in the time domain. Although the usefulness has been demonstrated through only two independent design problems; design of reduced-order robust model predictive control and input signal design for MIMO system identification, it is believed that the PRCM will extend its applications to more diverse MIMO control and identification problems.

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