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Relationships between Input-Output Stability and Exponentially Stable Periodic Orbits*

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Abstracts In this paper, we present new results concerning the relationship between the input-output and Lyapunov stability of nonlinear system possessing a periodic orbit. Definition of small-signal finite-gain L_p stability around periodic orbit is introduced. We show L_p stability of exponentially stable periodic orbit using quadratic Lyapunov functions for the periodic orbit. The L_2 gain analysis is presented with Hamiltonian-Jacobi inequality along with an example.

Keywords Nonlinear Systems, L_2 gain, Input-Output Stability, Periodic Orbits, Lyapunov Functions

1 Introduction

In this paper, we present new results concerning the relationship between the input-output and Lyapunov stability of nonlinear system possessing a periodic orbit. Both Lyapunov theory and input-output theory are well-developed for nonlinear system in [5, 7, 8, 9, 13, 16]. In Lyapunov stability it has been studied how Lyapunov stability can be used to establish L stability of nonliear systems around equilibrium point and the interrelationships between the two theories was presented [14]. We can expect the similar results on the stability on the periodic orbits. Although there are many results on stability of periodic orbits, most of them are concerning the stability of the periodic orbit in the state-space. On the other hand, the notion of input-output stability of periodic orbit is very appealing. This is because that we are often not able to get explicit solution of periodic orbit so that it is difficult to estimate the behaviour of orbit in the presence of external disturbances.

We have seen that many stability criteria regarding equilibirum point are easier to state and prove in the input-output stability than in the Lyapunov stability [5, 7, 8, 9, 13, 16]. However, many problems regarding periodic orbit stability are more naturally stated in the Lyapunov stability than in the input-output stability. Thus if one could conclude input-output stability from Lyapunov stability, it would be possible to use the techniques and results from the theory to solve the problems of the other. Thus, we study the relationship between the input-output and Lyapunov stability of periodic orbit.

This paper is organized as follows. We introduce local coordiantes, transverse dyanmics, and Converse Theorem in Section 2. In Section 3, L_p stability around periodic orbit is defined and relationships between input-output stability and Lyapunov stability of periodic orbit is presented. In Section 4, L_2 gain analysis is introduced and an example is shown to illustrate L_2 gain analysis. Then conclusion follows.

2 Preliminaries

The set of all n-dimensional vectors $x = (x_1, \cdots, x_n)^T$, where $x_1, \cdots, x_n$ are real numbers, defines the n-dimensional Euclidean space defined by $\mathbf{R}^n$. We shall consider the class of the p-norms, defined by

$$\|x\|_p = (|x_1|^p + \cdots + |x_n|^p)^{1/p}, \ 1 \le p < \infty$$

and

$$\|x\|_\infty = \max_i |x_i|$$

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The space L_p^n for $1 \le p < \infty$ is defined as the set of all piecewise continuous functions $x : [0, \infty) \to \mathbf{R}^n$ such that

$$\|x\|_{L_p^n} = \left(\int_0^\infty \|x(t)\|^p \, dt \right)^{1/p} < \infty$$

and the space L_∞^n is defined as the set of all piecewise continuous, uniformly bounded functions with the norm

$$\|x\|_{L_\infty^n} = \sup_t \|x(t)\| < \infty$$

Consider the smooth dynamical system

$$\dot{x} = f(x) \tag{1}$$

on $\mathbf{R}^n$ and suppose that $\eta \subset \mathbf{R}^n$ is a periodic orbit of (1) with period T.

Using the distance function

$$d(x, \eta) := \min_{y \in \eta} \|x - y\|$$

an ϵ-neighborhood of η can be specified as

$$B_\epsilon(\eta) := \{x \in \mathbf{R}^n : d(x, \eta) < \epsilon\}.$$

The orbit η is *stable* if trajectories starting near η stay near η, i.e., for every $\epsilon > 0$, there exists $\delta > 0$ such that $x \in B_\delta(\eta) \to \phi_t^f(x) \in B_\epsilon(\eta)$ for all $t \ge 0$ (ϕ_t^f is the *flow* of the vector field f); η is *asymptotically stable* if it is stable and trajectories starting near η converge to η, i.e., there is a $\delta > 0$ such that $d(\phi_t^f(x), \eta) \to 0$ as $t \to \infty$ for all $x \in B_\delta(\eta)$; η is *exponentially stable* if it is asymptotically stable and the convergence is exponential, i.e., there exist $\delta, M, \lambda > 0$ such that $d(\phi_t^f(x), \eta) \le d(x, \eta) M e^{-\lambda t}$ for all $x \in B_\delta(\eta)$. This type of stability is often called *orbital* stability to distinguish it from the often used concept of Lyapunov stability [4].

It is clear that the behavior that we are interested in is determined by the $n-1$ dimensional *transverse* dynamics. To highlight these dynamics, we construct a set of local coordinates $(\theta, \rho) \in S^1 \times \mathbf{R}^{n-1}$ as follows.

First, parameterizing S^1 by $\theta \in [0, T)$ (i.e., the closed segment $[0, T]$ with the two endpoints identified), define a mapping $\theta \mapsto y(\theta)$ by solving

$$\frac{dy(\theta)}{d\theta} = f(y(\theta))$$

with $y(0) \in \eta$ arbitrary (but fixed). Then, the θ coordinate of a point x in a neighborhood of η is defined by

$$\theta = \psi_1(x) := \arg \min_{\theta \in [0, T)} \|y(\theta) - x\|^2. \tag{2}$$

Note that the function $\psi_1(\cdot)$ is a smooth since $f(\cdot)$ is and the minimizer is unique on a small neighborhood of η. Next, letting

$v_i(\cdot)$, $i = 2, \ldots, n$, be a set of functions that are independent and vanish on η, the ρ coordinate of x is given by

$$\rho_{i-1} = v_i(x), \quad i = 2, \ldots, n. \tag{3}$$

The following proposition shows that it is always possible to find such functions.

Proposition 2.1 *Suppose that η is a periodic orbit of $\dot{x} = f(x)$. Then, there exist $n-1$ independent functions $\psi_i(\cdot)$, $i = 2, \ldots, n$ defined on a neighborhood of η that vanish on η.*

We now show that (θ, ρ) is a well defined set of coordinates in a neighborhood of η.

Proposition 2.2 *The mapping $x \mapsto (\theta, \rho)$ given by $(\theta, \rho) = \Psi(x)$ in (2) and (3) is a diffeomorphism on a neighborhood of η.*

Roughly speaking, since the ρ coordinates are *transverse* to the periodic orbit at each point of η, the stability of the orbit is largely determined by the behavior of the ρ coordinates. This is seen more clearly by examining the system dynamics in the (θ, ρ) coordinates.

Proposition 2.3 *The dynamics of the nonlinear system (1) in a neighborhood of the periodic orbit η have the form*

$$\begin{aligned} \dot{\theta} &= 1 + f_1(\theta, \rho) \\ \dot{\rho} &= A(\theta)\rho + f_2(\theta, \rho) \end{aligned} \tag{4}$$

where $f_1(\cdot, \cdot)$ and $f_2(\cdot, \cdot)$ satisfy

$$f_1(\theta, 0) = 0, \;\; f_2(\theta, 0) = 0, \;\; and \;\; \frac{\partial f_2(\theta, 0)}{\partial \rho} = 0.$$

The canonical form (4) provides a useful characterization of a periodic orbit:

Proposition 2.4 *The orbit η is an exponentially stable orbit of (1) if and only if the* transverse linearization

$$\frac{d\rho}{d\theta} = A(\theta)\rho \tag{5}$$

is asymptotically stable.

The linearization of a nonlinear system about an exponentially stable equilibrium point is commonly used in the construction of quadratic Lyapunov functions for the nonlinear system. Such functions are constructed by solving a Lyapunov equation. Roughly speaking, the *transverse linearization* takes the place of the equilibrium point linearization and a *periodic Lyapunov equation* takes the place of the Lyapunov equation.

Associated with the periodic linear system (5) is the *periodic Lyapunov equation*

$$P'(\theta) + A(\theta)^T P(\theta) + P(\theta)A(\theta) + Q(\theta) = 0 \tag{6}$$

where

$$P'(\theta) = \frac{dP}{d\theta}(\theta)$$

and $Q(\theta)$ is a continuous, periodic matrix. The following result is well known (see, e.g., [11], [1]).

Theorem 2.5 *Suppose that (5) is asymptotically stable and that $Q(\theta)$ is positive definite for all θ. Then (6) has a unique periodic solution $P(\theta)$ that is positive definite for all θ.*

We are now ready to construct quadratic Lyapunov functions that prove the exponential stability of periodic orbit η.

Theorem 2.6 (Converse Theorem) *[6] Suppose that η is an exponentially stable periodic orbit of (1). The Lyapunov function*

$$V(x) = \rho^T P(\theta)\rho$$

proves the exponential stability of η. That is, on a neighborhood of η, $V(\cdot)$ satisfies

$$k_1\|x\|_\eta^2 \le V(x) \le k_2\|x\|_\eta^2 \tag{7}$$

and

$$\left\| \frac{\partial V}{\partial x} \right\| \le k_3\|x\|_\eta \tag{8}$$

and

$$\dot{V}(x) \le -k_4\|x\|_\eta^2 \tag{9}$$

for some positive constants k_1, k_2, k_3, k_4 and

$$\|x\|_\eta := \inf_{y \in \eta} \|x - y\|.$$

3 L_p Stability and Input-Output Stability

Consider the nonlinear system

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned} \tag{10}$$

where $x \in \mathbf{R}^n$, $u \in \mathbf{R}^m$, $y \in \mathbf{R}^q$, f and the columns of g are smooth vector fields, and the elements of h are smooth functions. We may think of $x = (x_1, \cdots, x_n)$ as local coordinates for a smooth state space manifold of dimension n. Let η be a periodic orbit of (10) for $u = 0$ and suppose that h vanishes on η.

We consider the system (10) as a system whose input-output relation is represented by

$$y = Hu$$

where H is a mapping that specifies y in terms of u. With an extended space L_e^m defined as

$$L_e^m = \{u \mid u_\tau \in L^m, \; \forall \tau \ge 0\}$$

the mapping H is defined as a mapping from an extended space L_e^m to an extended space L_e^q. A mapping $H : L_e^m \to L_e^q$ is said to be causal if the value of the output $(Hu)(t)$ at any time t depends only on the values of the inputs up to time t. This is equivalent to

$$(Hu)_\tau = (Hu_\tau)_\tau$$

With the space of input and output signals defined as above, we can define input-output stability.

Definition 3.1 *A mapping $H : L_e^m \to L_e^q$ is finite-gain L stable (around η) if there exist nonnegative constants γ and β such that*

$$\|(Hu)_\tau\|_L \le \gamma\|u_\tau\|_L + \beta$$

for all $u \in L_e^m$ and $\tau \in [0, \infty)$.

The system is small-signal finite-gain L stable (around η) if it is true for a certain class of signal spaces L, i.e. small signal $u \in L_\infty$

Definition 3.2 *A mapping $H : L_e^m - L_e^q$ is small-signal finite-gain L stable (around η) if there exists a positive constant δ such that*

$$\|(Hu)_\tau\|_L \le \gamma\|u_\tau\|_L + \beta$$

is satisfied for nonnegative constant γ and β, and for all $u \in L_e^m$ with $\sup_{0 \le t \le \delta} \|u(t)\| \le \delta$.

If the nonlinear system (10) which has an exponentially stable closed orbit is disturbed, the ultimate boundedness of the solution of the disturbed system can be shown in the same way as the ultimate boundedness of solution of about an equilibirum point. If there are disturbances, we can no longer expect the solution of the disturbed system to approach to the periodic orbit as $t \to \infty$ nor the perturbed system to have the asymptotic phase. We will show uniform ultimate boundedness of exponentially stable periodic orbit subject to external disturbances. First, we construct a converse Lyapunov function for the exponentially stable periodic orbit. Then, we analyze its stability with respect to disturbance.

Theorem 3.1 *Consider a nonlinear system of the form*

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned} \quad (11)$$

where $x \in \mathbf{R}^n$, $u \in \mathbf{R}^m$, and $y \in \mathbf{R}^q$. f and the columns of g are smooth vector fields, and the elements of h are smooth functions. Suppose η is an exponentially stable periodic orbit of the unforced system of (11) such that

$$\dot{x} = f(x) \quad (12)$$

and $h(x)$ vanishes on η. Then the system (11) is small-signal finite-gain L_p stable around η for all $p \in [1, \infty]$.

Proof: By the converse theorem, there exists a C^1 Lyapunov function $V : U \to \mathbf{R}_+$ and positive constants k_1, k_2, k_3, and k_4 such that

$$k_1 \|x\|_\eta^2 \le V(x) \le k_2 \|x\|_\eta^2 \quad (13)$$

$$\left\| \frac{\partial V}{\partial x} \right\| \le k_3 \|x\|_\eta \quad (14)$$

$$\dot{V}_{(12)}(x) \le -k_4 \|x\|_\eta^2 \quad (15)$$

for $x \in B_r(\eta) \subset U$ where $\dot{V}_{(12)}(x)$ denotes the derivative of V along the trajectory of (12). Since $g(x)$ is bounded on $x \in U$, there exists l_g such that

$$\|g(x)\| \le l_g \quad \forall x \in U.$$

Furthermore, since $h(x)$ vanishes on η and is smooth, there exists l_h such that

$$\|y\| := \|h(x)\| \le l_h \|x\|_\eta \quad (16)$$

If we evaluate the derivative of V along the trajectory of (11), we get

$$\begin{aligned} \dot{V}_{(11)}(x) &= \frac{\partial V}{\partial x}(f(x) + g(x)u) \\ &\le -k_4 \|x\|_\eta^2 + k_3 l_g \|x\|_\eta \|u\| \end{aligned} \quad (17)$$

for $\|x\|_\eta < r$. Let $W = V^{\frac{1}{2}}$. Then W is continuously differentiable except at $x \in \eta$. Hence if $x \notin \eta$, we have $2W\dot{W} = \dot{V}$ so that (17) becomes

$$\begin{aligned} 2W\dot{W} &\le -k_4 \|x\|_\eta^2 + k_3 l_g \|x\|_\eta \|u\| \\ &\le -\frac{k_4}{k_2} W^2 + \frac{k_3 l_g}{\sqrt{k_1}} W \|u\|. \end{aligned} \quad (18)$$

Dividing both side of (18) by $2W$ gives

$$\dot{W} \le -\frac{k_4}{2k_2} W + \frac{k_3 l_g}{2\sqrt{k_1}} \|u\|$$

Now using comparison principle [15, 9] gives

$$W(t) \le e^{-t\frac{k_4}{2k_2}} W(0) + \frac{k_3 l_g}{2\sqrt{k_1}} \int_0^t e^{-(t-\tau)\frac{k_4}{2k_2}} \|u(\tau)\| u\tau$$

or

$$\|x\|_\eta \le \sqrt{\frac{k_2}{k_1}} \|x(0)\|_\eta e^{-t\frac{k_4}{2k_2}} + \frac{k_3 l_g}{2k_1} \int_0^t e^{-(t-\tau)\frac{k_4}{2k_2}} \|u(\tau)\| u\tau$$

It can be easily verified that

$$\|x(0)\|_\eta < r\sqrt{\frac{k_1}{k_2}} \quad \text{and} \quad \sup_{0 \le \sigma \le t} \|u(\sigma)\| < \frac{r k_1 k_4}{k_2 k_3 l_g}$$

ensures that $\|x(t)\|_\eta < r$. Hence $x(t)$ stays within $B_r(\eta)$. Using(16), we have

$$\|y\| \le c_1 e^{-at} + c_2 \int_0^t e^{-a(t-\tau)} \|u(\tau)\| d\tau \quad (19)$$

where

$$c_1 = \sqrt{\frac{k_2}{k_1}} \|x(0)\| l_h, \quad c_2 = \frac{k_3 l_g l_h}{2k_1}, \quad a = \frac{k_4}{2k_2}$$

Therefore, for $u \in L_{pe}^m$ and $\|u\|_{L_\infty^m} < \frac{r k_1 k_4}{k_2 k_3 l_g}$ for all $p \in [1, \infty)$, it can be easily verified that

$$\|y_\tau\|_{L_p} \le \gamma \|u_\tau\|_{L_p} + \beta$$

where

$$\gamma = \frac{c_2}{a}, \quad \beta = c_1 \delta$$

$$\delta = \begin{cases} 1, & \text{if } p = \infty \\ \left(\frac{1}{ap}\right)^{1/p}, & \text{if } p \in [1, \infty) \end{cases}$$

□

Remark We could get the same result with the relaxed conditions on $f(x)$, $g(x)$ and $h(x)$ such that $f(x)$ is continuously differentiable, $g(x)$ is bounded on a neighborhood U of η, $h(x)$ vanishes on the periodic orbit η and satisfies Lipschitz condition (16) around periodic orbit η.

Corollary 3.2 *In Theorem 3.1, if $u(\cdot) \in L_\infty^m$ and $u(t) \to 0$ as $t \to \infty$, then $y(t) \to 0$ as $t \to \infty$.*

Proof: It is shown that if $u(\cdot) \in L_\infty^m$ and $u(t) \to 0$ as $t \to \infty$ so does the convolution on the right side of (19) [3]. □

4 L_2 Gain Analysis

L_2 stability plays an important role in systems analysis since its square-integrable signals can be viewed as finite-energy signals. Nonlinear system (10) is represented as an input-output map from a disturbance u to the output y. We are often interested in finding not only the input-output stability but also the L_2 gain of the system. We showed how to calculate the L_2 gain for nonlinear system (10) in [2]. Consider the nonlinear system

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned} \quad (20)$$

where $x \in \mathbf{R}^n$, $u \in \mathbf{R}^m$, $y \in \mathbf{R}^p$. f and the columns of g are smooth vector fields, and the elements of h are smooth functions. Let η be a periodic orbit of (20) for $u = 0$ and suppose that h vanishes on η. A function V is *positive definite on η* if $V(x) = 0$, $x \in \eta$, and $V(x) > 0$ for $x \in B_\epsilon(\eta)\backslash\eta$ for some open neighborhood $B_\epsilon(\eta)$ of η. The nonlinear system (20) has *L_2 gain (around η) less than γ* if there is a constant $\delta > 0$ and $\epsilon > 0$ such that, for all $x_0 \in B_\epsilon(\eta)$, for all $u \in L_2$ with $\|u(t)\| \le \delta \ \forall t \in [0, \infty)$,

$$\int_0^\infty \|y(\tau)\|^2 d\tau < \gamma^2 \int_0^\infty \|u(\tau)\|^2 d\tau + 2V(x_0) \quad (21)$$

Sufficient conditions for showing that the L_2 gain (around η) of a system is less than γ look quite similar to the conditions for L_2 gain analysis around an equilibrium point [12].

Theorem 4.1 *[2] Consider the nonlinear system (10) and suppose that $V \in C^1$ is positive definite on η. We have the following implications:*

$$(A) \Leftrightarrow (B) \Rightarrow (C)$$

(A) There is an open neighborhood $B_\epsilon(\eta)$ of η such that

$$\frac{\partial V}{\partial x}(x)f(x) + \frac{1}{2\gamma^2}\frac{\partial V}{\partial x}(x)g(x)g^T(x)\frac{\partial^T V}{\partial x}(x) + \frac{1}{2}h^T(x)h(x) < 0 \quad (22)$$

for $x \in B_\epsilon(\eta) \setminus \eta$.

(B) There is an open neighborhood $B_\epsilon(\eta)$ of η such that

$$\frac{\partial V}{\partial x}(x)f(x) + \frac{\partial V}{\partial x}(x)g(x)u < \frac{1}{2}\gamma^2\|u\|^2 - \frac{1}{2}\|h(x)\|^2 = \frac{1}{2}\gamma^2\|u\|^2 - \frac{1}{2}\|y\|^2 \quad (23)$$

for $x \in B_\epsilon(\eta) \setminus \eta$ and $u \in \mathbf{R}^m$.

(C) The system has L_2 gain less than γ around η.

Inequality (22) is known as the Hamilton-Jacobi inequality. We showed that, given an exponentially stable periodic orbit, a continously differentiable $V(x)$ can be constructed using the local coordinates presented in Section 2. Unlike Theorem 3.1, in Theorem 4.1 we do not require the periodic orbit of the unforced system to be exponentially stable. This is illustrated by the following examples.

Example Consider the planar system

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned} \tag{24}$$

where

$$f(x) = \begin{bmatrix} -(\sqrt{x_1^2 + x_2^2} - 1)^3 \frac{x_1}{\sqrt{x_1^2 + x_2^2}} - x_2 \\ -(\sqrt{x_1^2 + x_2^2} - 1)^3 \frac{x_2}{\sqrt{x_1^2 + x_2^2}} + x_1 \end{bmatrix},$$

$$g(x) = \begin{bmatrix} 0 \\ \sqrt{x_1^2 + x_2^2} - 1 \end{bmatrix}, \quad h(x) = \left(\sqrt{x_1^2 + x_2^2} - 1\right)^2$$

Take Σ as our cross section such that

$$\Sigma = \{(x_1, x_2) \in \mathbf{R}^2 \mid x_1 > 0, x_2 = 0\}.$$

Transforming the system (24) to polar coordinates (r, θ) where $r = (x_1^2 + x_2^2)^{1/2}$, $\theta = \arctan(x_2/x_1)$, we obtain

$$\begin{aligned} \dot{r} &= -(r-1)^3 \\ \dot{\theta} &= 1. \end{aligned} \tag{25}$$

Then the cross section becomes

$$\Sigma = \{(r, \theta) \in \mathbf{R}_+ \times S^1 \mid r > 0, \theta = 0\}$$

and the Poincaré map is given by

$$P_{(25)}(r_0) = 1 + \frac{r_0 - 1}{\sqrt{1 + 4\pi(r_0 - 1)^2}}.$$

Clearly, the Poincaré map has a fixed point at $r_0 = 1$, reflecting the circular closed orbit of radius 1 of the polar system. The linearization of the Poincaré map is given by

$$\begin{aligned} DP_{(25)}(1) &= \left.\frac{dP}{dr_0}\right|_{r_0=1} = \left.\frac{1}{[1 + 4\pi(r_0 - 1)^2]^{3/2}}\right|_{r_0=1} \\ &= 1. \end{aligned} \tag{26}$$

We note easily the periodic orbit of the example is asymptotically stable. But the derivative of the Poincaré map is equal to 1 so that we cannot make a conclusion on the stability of the planar system from the Poincaré map. Notice that linearized Poincaré map give us only sufficient condition to asymptotic stability. In addition, the construction of the Poincaré map relies on prior knowledge of the solution of the differential equation. Therefore, except for trivial examples where the solution is available in a closed form, it is very difficult to construct the Poincaré map analytically. In many cases, it can only be computed numerically [10].

In the example, we use $V(x) = \frac{1}{2}(\sqrt{x_1^2 + x_2^2} - 1)^2$ as $V(x)$ to show that the system has an asymptotically stable periodic orbit. Using $V(x) = \frac{1}{2}(\sqrt{x_1^2 + x_2^2} - 1)^2$ as a Lyapunv candidate function for the solution of the Hamiltonian-Jacobi inequality (22), it can be shown that

$$\begin{aligned} &\frac{\partial V}{\partial x}(x)f(x) + \frac{1}{2\gamma^2}\frac{\partial V}{\partial x}(x)g(x)g^T(x)\frac{\partial^T V}{\partial x}(x) + \frac{1}{2}h^T(x)h(x) \\ =\ & -\frac{1}{2}\left(\sqrt{x_1^2 + x_2^2} - 1\right)^4 + \frac{1}{2\gamma^2}\left(\sqrt{x_1^2 + x_2^2} - 1\right)^4 \\ =\ & -\frac{1}{2}\left(1 - \frac{1}{\gamma^2}\right)\left(\sqrt{x_1^2 + x_2^2} - 1\right)^4 < 0 \\ & \forall x \in B_\epsilon(\eta)/\eta \text{ and } \gamma > 1 \end{aligned}$$

Thus this system is small-signal finite-gain L_2 stable and its L_2 gain is less than γ $(\gamma > 1)$. We note that the condition of Theorem 3.1 are not satisfied in this example since the periodic orbit of the unforced system is not exponentially stable. This example shows that we could still calculate L_2 gain using the Hamiltonian-Jacobi Inequality (22), although the system does not have exponentially stabel periodic orbit.

5 Conclusion

In this paper, we presented new results concerning the relationship between the input-output and Lyapunov stability of nonlinear system possessing a periodic orbit. Definition of small-signal finite-gain L_p stability around periodic orbit was introduced. Furthermore, we showed L_p stability of exponentially stable periodic orbit using quadratic Lyapunov functions for the periodic orbit. Finally, the L_2 gain analysis was presented with Hamiltonian-Jacobi inequality along with an example.

References

[1] Sergio Bittanti, Patrizio Colaneri, and Giuseppe De Nicolao. The periodic riccati equation. In S. Bittani, A. Laub, and J. C. Willems, editors, *The Riccati Equation*, chapter 6, pages 127–162. Springer Verlag, 1991.

[2] Chung Choo Chung and John Hauser. Nonlinear H_∞ control around periodic orbits. *Systems and Control Letters*, 30:127–137, 1997.

[3] C. A. Desoer and M. Y. Wu. Stability of linear Time-Invariant systems. *IEEE Transactions on Circuit Theory*, 15:245–250, Sept 1968.

[4] W. Hahn. *Stability of Motion*. Springer-Verlag, Berlin, 1967.

[5] Jack K. Hale. *Ordinary Differential Equations*. Robert E. Krieger Publishing Company, Malabar, Florida, 1980.

[6] John Hauser and Chung Choo Chung. Converse Lyapunov functions for exponentially stable periodic orbits. *Systems and Control Letters*, 23:27–34, 1994.

[7] Hassan K. Khalil. *Nonlinear Systems*. Prentice Hall, New Jersey, 1996. Second Edition.

[8] N. N. Krasovskii. *Stability of Motion*. Stanford University Press, Stanford, 1963.

[9] Richard K. Miller and Anthony N. Michel. *Ordinary Differntial Equations*. Academic Press, New York, 1982.

[10] Thomas S. Parker and Leon O. Chua. *Practical Numerical Algorithms for Chaotic Systems*. Springer-Verlag, New York, 1989.

[11] Mark A. Shayman. Inertia theorems for the periodic Liapunov equation and periodic Riccati equation. *Systems and Control Letters*, 4:27–32, 1984.

[12] Arjan J. van der Schaft. L_2-gain analysis of nonlinear systems and nonlinear state feedback H_∞ control. *IEEE Transactions on Automatic Control*, 37(6):770–784, 1992.

[13] M. Vidyasagar. *Nonlinear Systems Analysis*. Prentice-Hall, Inc., Englewood Cliffs, N.J., 1993. Second Edition.

[14] M. Vidyasagar and A. Vanelli. New relationships between input-output and Lyapunov stability. *IEEE Transactions on Automatic Control*, 27:481–483, 1982.

[15] Wolfgang Walter. *Differential and Integral Ineqaulities*. Springer-Verlag, New York, 1970.

[16] Taro Yoshizawa. *Stability Theory by Lyapunov's Second Method*. The Mathematical Society of Japan, Japan, 1966.