The Advanced z-Transform and Analysis of Sampled-Data Systems

Tae-Sang Chung
Dept. of Control and Instrumentation Engineering
Chung-Ang University

Abstract - The z-transform method is a basic mathematical tool in analyzing and designing sampled-data control systems. However, since the z-transform method relates only the sampling-instants signals, another mathematical tool is necessary to describe the continous signals between the sampling instants. For this purpose the delayed and the modified z-transform methods were developed. The definition of the modified z-transform includes a sample in the interval $[-T, 0]$ of the original signal in its series expression, where the signal value is always zero for any physical system. From this reason one step skew of the time index always appears in its application formulas. This introduces an unnecessary operation and a gap in linking the mathematical formula and its physical interpretation. Considering the conceptual difficulty and application inconvenience, a method of using the *advanced z-transform* in analysis of sampled-data control systems is developed as a replacement of the modified z-transform. With one formulation of the advanced z-transform, now it is possible to relate both the signals of the sampling instants and those in between without any complication and conceptual difficulty.

I. Introduction

In a sampled-data control system, the input and output signals of the digital block and the input signals to the plant are only updated at the sampling instants. However, the internal state variables including the plant outputs are still continuous signals except possibly jump discontinuities at the sampling instants. While the z-transform is used to represent the relation of the values of these signals at the sampling instants, the modified z-transform method can be used to relate a continuous signal at the off sampling instants to those signals at the sampling instants [1-5].

Fig. 1 is a basic block of sampled-data control systems, which includes an input $E(s)$, a sampler with sampling time T, a plant $G(s)$, and its output $C(s)$. The inverse Laplace transform of $G(s)$ is the plant's impulse response and is denoted by $g(t)$. When the sampler is assumed to be an ideal impulse sampler and the plant $G(s)$ is also assumed to respond to the impulse, the output $c(t)$ at time $t=kT$ can be obtained by the following discrete convolution:

$$c(kT)=\sum_{j=0}^{k} g(kT-jT)e(jT)=\sum_{j=0}^{\infty} g(kT-jT)e(jT) \qquad (1)$$

where $g(t)=0$ for $t<0$ was used in extending the upper limit of the running variable j to infinity. The z-transform of the output $c(t)$ is defined as

$$\begin{aligned} C(z) &= Z[C(s)]=Z[c(t)]=Z[c(kT)] \\ &= \sum_{k=0}^{\infty} c(kT)z^{-k} \\ &= c(0)+c(T)z^{-1}+\cdots+c(kT)z^{-k}+\cdots \end{aligned} \qquad (2)$$

Substituting (1) into (2) gives the expression of the z-transform of $c(t)$ in terms of those of the input $e(t)$ and the plant $g(t)$ [1-4]:

$$\begin{aligned} C(z) &= \sum_{k=0}^{\infty} c(kT)z^{-k} \\ &= \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} g(kT-jT)e(jT)z^{-k}=G(z)E(z) \end{aligned} \qquad (3)$$

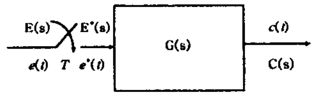

Fig. 1. A sampled-data control system. The sampler is an ideal impulse sampler with sampling time T. The plant $G(s)$ is assumed to respond to the impulse.

Once $G(z)$ and $E(z)$ are evaluated, $C(z)$ is obtained by (3), from which the time sequence c_k or $c(kT)$ with T being understood as the sampling time can be evaluated by the inverse z-transform operation:

$$c_k=c(kT)=Z^{-1}[C(z)]=Z^{-1}[G(z)E(z)] \qquad (4)$$

The above relation gives the output $c(t)$ only at the sampling instants, and it does not give any detail of the behavior of $c(t)$ between the sampling instants.

To handle the response of a continuous signal between the sampling instants, the delayed z-transform and the modified z-transform were developed [5-9]. The modified z-transform of a continuous signal $c(t)$ is defined as the z-transform of the original signal delayed by $(1-m)T$:

$$\begin{aligned} Z_m[c(t)] &= C(z,m)=\sum_{k=0}^{\infty} c(kT-T+mT)z^{-k} \\ &= c(-T+mt)+c(mt)z^{-1}+c(T+mt)z^{-2}+\cdots \end{aligned} \qquad (5)$$

where $0<m<1$. Since $c(t)=0$ for $t<0$, (5) can be written as

$$\begin{aligned} Z_m[c(t)] &= C(z,m)=z^{-1}\sum_{k=0}^{\infty} c(kT+mT)z^{-k} \\ &= c(mT)z^{-1}+c(T+mT)z^{-2}+\cdots \\ &\quad +c(kT-T+mT)z^{-k}+\cdots \end{aligned} \qquad (6)$$

Multiplying both sides of (6) by z gives

$$zC(z,m)=\sum_{k=0}^{\infty} c(kT+mT)z^{-k} \qquad (7)$$

and its inverse z-transform gives

$$c(kT+mT)=Z^{-1}[zC(z,m)] \qquad (8)$$

which is $c(t)$ for the time duration $kT\le t<kT+T$, $k\ge 0$ and when m is varied between zero and one.

Consider the system in Fig. 1 again. The output $c(t)$ at time $t=kT+mT$ is expressed by the discrete convolution:

$$\begin{aligned} c(kT+mT) &= \sum_{j=0}^{k} g(kT+mT-jT)e(jT) \\ &= \sum_{j=0}^{\infty} g(kT+mT-jT)e(jT) \end{aligned} \qquad (9)$$

Substituting (9) into (6) and performing some algebraic manipulation gives [1-4]

$$\begin{aligned} C(z,m) &= z^{-1}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty} g(kT+mT-jT)e(jT)z^{-k} \\ &= E(z)G(z,m) \end{aligned} \qquad (10)$$

where $G(z,m)$ is the modified z-transform of $g(t)$. Once $G(z,m)$ and $E(z)$ are found, $C(z,m)$ is obtained by (10), and substituting the result into (8) gives

$$c(kT+mT)=Z^{-1}[zG(z,m)E(z)] \qquad (11)$$

As given in (10) and (11), the modified z-transform method is straightforward in describing a continuous signal of any sampled-data system between the sampling instants. However, as shown in (6) and (8) or (11), its related formulas contain z or z^{-1} term in their expressions, representing advance or delay of one sampling interval. Careful observation of (11) reveals that the term z is multiplied to $G(z,m)E(z)$ before taking the inverse z-transform operation for $c(kT+mT)$ to cancel the term z^{-1} which appears as a factor in the expression of $G(z,m)$ which is defined by the same way as given in (6). If the extra z or z^{-1} term is properly taken care of, this is not a problem in the mathematical point of view. However, this introduces a double folded gap in linking the mathematical formula and its physical interpretation, and also introduces inconvenience in handling the signals in the z domain.

Notice that, as given in (5), the definition of the modified z-transform of the $c(t)$ includes $c(-T+mT)$, the function value at $t=-T+mt$, although it is zero. The meaning of the modified z-transform transfer relation established in (10) is that it relates the input sequence applied to the plant from $t=0$ to the output sequence sampled periodically starting from $t=-(T-mT)<0$ through the transfer function $G(z,m)$. The input at and after $t=0$ can cause an effect on the output at and after $t=0$ at best. Therefore it can be doubted that, strictly speaking, the concept of the transfer relation of (10) violates the cause and effect rule associated with physical plants. The reason that (10) is still valid is that $c(t)$ at $t=-T+mt<0$ due to input $e(t)$ at and after $t\geq 0$ is set to zero (using $g(t)=0$ for $t<0$), and actually does not appear in its modified z-transform series expression, eliminating the potential noncausality problem. However, as pointed out in the previous paragraph, the term z^{-1} or z term remains in the related formulas of the modified z-transform transfer relation.

II. The Advanced z-Transform and Analysis of Sampled-Data Control Systems

Considering the conceptual difficulty and application inconvenience of the modified z-transform method, another variation of the z-transform method which relates the input at and after $t=0$ to the output at and after $t=0$ is developed in this paper.

Consider the following variation of the z-transform of a continuous signal $c(t)$:

$$\begin{aligned} Z_a[c(t)] &= C(z,a) = Z[c(kT+aT)] \\ &= \sum_{k=0}^{\infty} c(kT+aT)z^{-k} \\ &= c(aT) + c(T+aT)z^{-1} + \cdots \\ &\quad + c(kT+aT)z^{-k} + \cdots \end{aligned} \tag{12}$$

where T is the sampling time and $0\leq a<1$. Since the above definition is the z-transform of function $c(t+aT)$ which is $c(t)$ advanced by aT in the time axis, the above z-transform is called the *advanced z-transform* of $c(t)$. Then, the inverse relation of the advanced z-transform becomes

$$c(kT+aT) = Z^{-1}[C(z,a)] \tag{13}$$

With $a=0$, the signal is not advanced and the advanced z-transform becomes the regular z-transform, or

$$C(z,a)|_{a=0} = C(z,0) = C(z) \tag{14}$$

The advanced z-transform of a typical signal $g(t)=e^{-at}$ becomes

$$\begin{aligned} G(z,a) &= Z_a\left[\frac{1}{s+a}\right] = Z_a[e^{-at}] \\ &= Z[e^{-a(t+aT)}] = e^{-aaT}Z[e^{-at}] = e^{-aaT}\frac{z}{z-e^{-aT}} \end{aligned} \tag{15}$$

Note that the advanced z-transform of e^{-at} is the z-transform of e^{-at} itself multiplied by e^{-aaT}, which carries the shape of the original signal variation in the interval $0\leq t<T$. It should be mentioned, however, that this fashion is individually applicable to a single mode component signal. The formulas of the advanced z-transform of the practical engineering signals can be derived easily based on the above formulation or simply multiplying z to the formulas of the modified z-transform if available. For example, the advanced z-transform of the unit step function, $u_s(t)$, can be obtained by setting $a=0$ in (15):

$$Z_a\left[\frac{1}{s}\right] = Z_a[u_s(t)] = Z[u_s(t+aT)] = \frac{z}{z-1} \tag{16}$$

The advanced z-transform of $\sin wt$ is derived as

$$\begin{aligned} Z_a[\sin wt] &= Z[\sin w(t+aT)] \\ &= \cos waT\, Z[\sin wt] + \sin waT\, Z[\cos wt] \\ &= \frac{z(z\sin waT + \sin w(1-a)T)}{z^2-2z\cos wT+1} \end{aligned} \tag{17}$$

Considering function $\sin wt$ to contain two modes, $-jw$ and jw, the above result can also be derived as follows:

$$\begin{aligned} Z_a[\sin wt] &= Z_a\left[\frac{1}{2j}(e^{jwt}-e^{-jwt})\right] \\ &= \frac{1}{2j}(e^{jwaT}Z[e^{jwt}] + e^{-jwaT}Z[e^{-jwt}]) \end{aligned} \tag{18}$$

In order to establish a transfer relation using the advanced z-transform method, first $c(kT+aT)$ needs to be expressed in terms of $g(t)$ and $e(t)$, which is the same as (9) when m is replaced with a:

$$\begin{aligned} c(kT+aT) &= \sum_{j=0}^{k} g(kT+aT-jT)e(jT) \\ &= \sum_{j=0}^{\infty} g(kT+aT-jT)e(jT) \end{aligned} \tag{19}$$

Taking the advanced z-transform on both sides of (19) gives

$$\begin{aligned} C(z,a) &= \sum_{k=0}^{\infty} c(kT+aT)z^{-k} \\ &= \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} g(kT+aT-jT)e(jT)z^{-k} \\ &= \sum_{j=0}^{\infty} e(jT)z^{-j}\sum_{k=0}^{\infty} g(kT+aT-jT)z^{-(k-j)} \\ &= \sum_{j=0}^{\infty} e(jT)z^{-j}\sum_{k-j=0}^{\infty} g(kT-jT+aT)z^{-(k-j)} \\ &= \sum_{j=0}^{\infty} e(jT)z^{-j}G(z,a) = E(z)G(z,a) \end{aligned} \tag{20}$$

The above relation is the fundamental theoretical background to make it possible to establish the z-transform transfer equation to relate the values of a continuous signal at time being apart by aT from every sampling instant to the values of other signals, either discrete or continuous, at the sampling-instants.

Consider the system in Fig. 1 again. If $G(s)=1/(s+1)$ and $E(s)=1/s$, then $C(z,a)$ is evaluated by substituting (15) and (16) into (20):

$$C(z,a) = G(z,a)E(z) = e^{-aaT}\frac{z}{z-e^{-aT}}\,\frac{z}{z-1} \tag{21}$$

The output $c(t)$ between the sampling instants can now be obtained by taking the inverse z-transform of (21):

$$c(kT+aT) = Z^{-1}\Big[\frac{z^2}{(z-e^{-aT})(z-1)}\Big]e^{-aaT} = \frac{1-e^{-a(k+1)T}}{1-e^{-aT}}\, e^{-aaT} \qquad (22)$$

where $k \geq 0$ and $0 \leq a < 1$. Setting $a=0$ (no advance) in (21) and (22) gives, respectively,

$$C(z) = C(z,a)|_{a=0} = \frac{z^2}{(z-e^{-aT})(z-1)} \qquad (23)$$

and

$$c(kT) = \frac{1-e^{-a(k+1)T}}{1-e^{-aT}}\, e^{-aaT}\Big|_{a=0} = \frac{1-e^{-a(k+1)T}}{1-e^{-aT}} \qquad (24)$$

With the time index k taking integer values from 0 and a being any fractional value between 0 and 1, $c(t)$ for any t including the sampling instants can be evaluated using (22). Fig. 2 shows the response of $c(t)$ for $T=1$ and $a=1$. As given in (24) and as shown in Fig. 2, when t increases from $(k-1)T$ to kT, $c(t)$ increases by e^{-akT}, and in this pattern, $c(kT)$ approaches $1/(1-e^{-aT})$ exponentially as k approaches infinity. However, neither the behavior of $c(t)$ between the sampling instants nor the continuities at the sampling instants can be indicated by $C(z)$ of (23) or $c(kT)$ of (24).

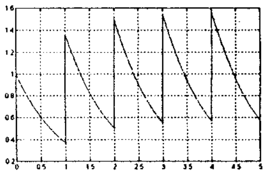

Fig. 2. Output response of the sampled-data system in Fig. 1 for the unit impulse train input. The response shows jump discontinuities at the sampling instants.

In the practical sampled-data control systems, the digitally processed sampled signals are applied to the plant through a zero order hold device (ZOH). When a ZOH is embedded between the sampler and the plant, the advanced z-transform of $c(t)$ of Fig. 1 becomes

$$\begin{aligned} C(z,a) &= Z_a\Big[\frac{1-e^{-Ts}}{s}G(s)\Big] = Z\Big[\frac{1-e^{-Ts}}{s}G(s)e^{aTs}\Big] \\ &= (1-z^{-1})Z\Big[\frac{G(s)}{s}e^{aTs}\Big] \\ &= (1-z^{-1})Z_a\Big[\frac{G(s)}{s}\Big] \end{aligned} \qquad (25)$$

The advanced z-transform method can be applied to closed-loop sampled-data systems. Consider a simple closed loop system given in Fig. 3. The z-transform and the advanced z-transform can be applied to the discrete signals and the continuous signals, respectively:

$$C(z,a) = G(z,a)E(z) \qquad (26)$$

$$E(z,a) = R(z,a) - GH(z,a)E(z) \qquad (27)$$

$$E(z) = R(z) - GH(z)E(z) \qquad (28)$$

Solving for $E(z)$ from (28) and substituting the result in (26) and (27) gives, respectively,

$$C(z,a) = \frac{G(z,a)}{1+GH(z)}R(z) \qquad (29)$$

$$E(z,a) = R(z,a) - \frac{GH(z,a)}{1+GH(z)}R(z) \qquad (30)$$

The output and the error responses between the sampling instants can be obtained by taking the inverse z-transforms on (29) and (30), respectively:

$$c(kT+aT) = Z^{-1}\Big[\frac{G(z,a)}{1+GH(z)}R(z)\Big] \qquad (31)$$

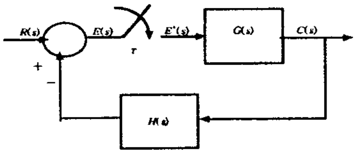

Fig. 3. A colsed-loop sampled-data control system

$$e(kT+aT) = Z^{-1}\Big[R(z,a) - \frac{GH(z,a)}{1+GH(z)}R(z)\Big] \qquad (32)$$

III. Conclusions

In this paper a new variation of the z-transform which removes the z^{-1} term of the modified z-transform was introduced. While it is not possible to relate the present input to the past output, relating the present input to the future output is compliant to the principle of cause and effect. Since there is no time index skew, the expression of the new z-transform is self explanatory, removing the conceptual gap between the behavior of the sampled-data system and its mathematical model which was observed with the modified z-transform. Therefore the advanced z-transform is physically and mathematically more natural than the modified z-transform.

Although the advanced z-transform does not give any additional information than the modified z-transform does, with the new method, one mathematical formulation describes directly both the sampling-instant signals and a continuous signal at off the sampling instants. Also the new method is more straightforward in its derivation. The theorems established with the modified z-transform for analyzing sampled-data control systems are equally effective to the advanced z-transform with minor modifications. As a conclusion, the advanced z-transform is a good replacement of the modified z-transform.

IV. References

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