Modified z-변환의 성질상의 해석적 문제점

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Analytic Problems of the Properties of the Modified z-Transform

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Abstract - While the z-transform method is a basic mathematical tool to relate the signals only at the sampling instants in analyzing and designing sampled-data control systems, the modified z-transform which is a variation of the z-transform is widely used to represent the details of continuous signals between the sampling instants. Regarding the modified z-transform method, some properties were established to relate the modified z-transform to the regular z-transform. This paper will show that these properties, in their current forms, cause some analytic problems, when they are applied to the signals with discontinuities at the sampling instants, which accordingly limit their applications significantly. In this paper, those analytic problems will be investigated, and the theorems of the modified z-transform will be revised by adopting a new notation so that those can be correctly interpreted and used without any analytic problems in the analysis of sampled data systems. Also some useful schemes of applying the modified z-transform will be developed.

I. Introduction

In the sampled-data control systems, some of the continuous signals of the controlled plant are sampled periodically and those sampled values are manipulated through control-law-based digital computation and applied to the plant and held constant up to the next sampling instant. To represent the relation of these signal values at the sampling instants, the z-transform method is used.

The definition of the z-transform of a continuous signal $c(t)$ with the sampling time T [1-4] is given below for the purpose of cross reference and comparison with the new development in this paper:

$$\begin{aligned} C(z) &= Z[C(s)] = Z[c(t)] = Z[c(kT)] = \sum_{k=0}^{\infty} c(kT)z^{-k} \\ &= c(0) + c(T)z^{-1} + \ldots + c(kT)z^{-k} + \ldots \end{aligned} \quad (1)$$

where $C(s)$ is the Laplace transform of $c(t)$. Referring to (1), the z-transform of a typical engineering signal $g(t) = e^{-at}$ becomes

$$G(z) = Z\left[\frac{1}{s+a}\right] = Z[e^{-at}] = \sum_{k=0}^{\infty} e^{-akT}z^{-k} = \frac{z}{z-e^{-aT}} \quad (2)$$

Since the z-transform takes the values of a continuous signal only at the sampling instants, it is not possible to recover from its z-transform the exact behavior of the original function between the sampling instants. The inverse z-transform only gives the kth term c_k of the periodic samples of the original function $c(t)$, or the function value at time kT, $c(kT)$, if T is understood as the sampling period. When the notation of the inverse z-transform is given as

$$c_k = c(kT) = Z^{-1}[C(z)], \quad (3)$$

the counterpart of (2) becomes

$$g_k = g(kT) = Z^{-1}\left[\frac{z}{z-p}\right] = p^k \quad (4)$$

Fig. 1 shows a basic block of sampled-data control systems. It is assumed that the sampler is an ideal impulse sampler with sampling time T. The plant $G(s)$ is also assumed to respond to the impulse. The impulse response of the plant $g(t)$ is the inverse Laplace transform of $G(s)$. When the input $e(t)$ is impulse-sampled and applied to the plant, the output $c(t)$ at time $t=kT$ can be obtained by the following discrete convolution:

$$c(kT) = \sum_{j=0}^{k} g(kT-jT)e(jT) = \sum_{j=0}^{\infty} g(kT-jT)e(jT) \quad (5)$$

In (5), the causal property that $g(t)=0$ for $t<0$ was used in expanding the upper limit of the running variable j up to infinity. Taking the z-transform of both sides of (5) yields [1-4]

$$\begin{aligned} C(z) &= \sum_{k=0}^{\infty} c(kT)z^{-k} \\ &= \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} g(kT-jT)e(jT)z^{-k} = G(z)E(z) \end{aligned} \quad (6)$$

The relation (6) makes it possible to use the z-transform method in formulating the pulse transfer function in the sampled-data system.

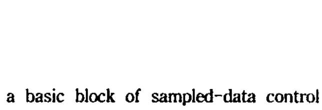

Fig. 1. A sampled-data control system. The sampler is an ideal impulse sampler with sampling time T. The plant $G(s)$ is assumed to respond to the impulse.

As an illustrative example, consider the case that the input $E(s) = 1/s$ and the plant $G(s) = 1/(s+a)$. Applying the z-transform to the input gives

$$E(z) = Z\left[\frac{1}{s}\right] = \sum_{k=0}^{\infty} z^{-k} = \frac{z}{z-1} \quad (7)$$

and $G(z)$ was evaluated already and given in (2). Substituting (2) and (7) into (6) gives the z-transform of the output:

$$\begin{aligned} C(z) &= \frac{z^2}{(z-1)(z-e^{-aT})} \\ &= \frac{1}{1-e^{-aT}}\left[\frac{z}{z-1} - \frac{e^{-aT}z}{z-e^{-aT}}\right] \end{aligned} \quad (8)$$

Applying the inverse z-transform formula of (4) to (8) yields the output sequence:

$$\begin{aligned} c_k &= c(kT) = \frac{1}{1-e^{-aT}}[1-e^{-aT}e^{-kaT}], \\ &= 1 + e^{-aT} + e^{-2aT} + \ldots + e^{-kaT} \end{aligned} \quad (9)$$

As given in (9), when t increases from $(k-1)T$ to kT,

$c(t)$ increases by e^{-akT}, and in this pattern, $c(kT)$ approaches $1/(1-e^{-aT})$ exponentially as k approaches infinity. However, neither the behavior of $c(t)$ between the sampling instants nor the continuities at the sampling instants can be obtained by (8) or (9).

II. The Right and Left z-Transforms

While the input and output signals of the digital block and the input to the plant are only updated at the sampling instants, the internal state variables including the plant outputs are still continuous signals except possibly jump discontinuities at the sampling instants.

Since the system in Fig. 1 is of first order and the unit impulse is applied to the plant every T sec, the response of the system during the interval $kT \le t < kT+T$ can be obtained by applying the superposition theorem on the delayed impulse responses of $g(t-jT)=e^{-a(t-jT)}$, $j \ge 0$. Then $c(t)$ becomes

$$c(t) = \sum_{j=0}^{k} g(t-jT) = \sum_{j=0}^{k} e^{-a(t-jT)} = \frac{(1-e^{-a(k+1)T})}{1-e^{-aT}} e^{-a(t-kT)} \quad (10)$$

for $kT \le t < kT+T$ and $k \ge 0$. Fig. 2 shows the response of $c(t)$ with $T=1$ and $a=1$. As shown in Fig. 2, $c(t)$ has jump discontinuities at the sampling instants kT.

Substituting $k-1$ for k in (10) gives

$$c(t) = \frac{(1-e^{-akT})}{1-e^{-aT}} e^{-aT} e^{-a(t-kT)} \quad (11)$$

for $kT-T \le t < kT$ and $k \ge 1$. Since the system is considered causal, $c(t)=0$ for $t<0$ for both of (10) and (11). Setting $t=kT$ in both (10) and (11) gives, respectively,

$$c(kT^{+}) = \frac{(1-e^{-a(k+1)T})}{1-e^{-aT}} \quad (12)$$

and

$$c(kT^{-}) = \frac{(1-e^{-akT})}{1-e^{-aT}} e^{-aT} \quad (13)$$

where $t=kT^{+}$ and $t=kT^{-}$ mean that the time variable t approaches kT from the right-hand side and the left-hand side on the time axis, respectively. The jump d_k at the sampling instant at $t=kT$, $k=1,2,\ldots,$ can be obtained by subtracting (13) from (12):

$$d_k = d(kT) = c(kT^{+}) - c(kT^{-}) = 1, \quad k \ge 1 \quad (14)$$

The jump at $t=0$ is also 1 since $c(0^{+})=1$ from (12) and $c(0^{-})=0$.

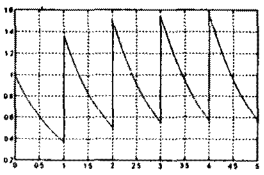

Fig. 2. Output response of the sampled-data system in Fig. 1. for the unit impulse train input. The response shows jump discontinuities at the sampling instants.

Comparing (12) with (9) reveals that $c(kT)$ obtained by taking the inverse z-transform of $C(z)$ in (8) and accordingly that used in (6) for defining $C(z)$ actually means $c(kT^{+})$ and not $c(kT^{-})$, which has been common understanding in this field. By the same argument, $g(kT)$ in (6) represents $g(kT^{+})$. This clarification takes an important role in developing the material of this paper in regarding the modified z-transform [5-8]. For the purpose of comparison and easy cross reference for the material developed in this paper, the z-transform of $c(kT^{-})$ of (13) is evaluated below. This z-transform is defined as the *left z-transform* and is denoted $C^{-}(z)$:

$$C^{-}(z) = \sum_{k=0}^{\infty} c(kT^{-}) z^{-k} = \sum_{k=1}^{\infty} \frac{(1-e^{-akT})}{1-e^{-aT}} e^{-aT} z^{-k} = \frac{e^{-aT} z}{(z-1)(z-e^{-aT})} \quad (15)$$

where $c(0^{-})=0$ was used. Similarly the *right z-transform* $C^{+}(z)$ is defined as the z-transform of $c(kT^{+})$, and it turns out that $C^{+}(z)$ actually means $C(z)$ in (8). If a signal does not have jump discontinuity at any sampling instant, then its left z-transform and the right z-transform are the same, otherwise the two transforms are different from each other.

III. Analytical Problems and Revisions of the Properties of the Modified z-Transform

To handle the response of a continuous signal between the sampling instants, variations of z-transform were developed [5-9]. The modified z-transform is defined as

$$Z_m[c(t)] = C(z,m) = \sum_{k=0}^{\infty} c(kT-T+mT) z^{-k} \quad (16)$$

where $0<m<1$ and the signal is delayed by $(1-m)T$. Since $c(t)=0$ for $t<0$, (16) can be written as

$$Z_m[c(t)] = C(z,m) = z^{-1} \sum_{k=0}^{\infty} c(kT+mT) z^{-k} \quad (17)$$

Consider the system in Fig. 1 again. The output $c(t)$ at time $t=kT+mT$ is expressed by the discrete convolution:

$$c(kT+mT) = \sum_{j=0}^{k} g(kT+mT-jT) e(jT) = \sum_{j=0}^{\infty} g(kT+mT-jT) e(jT) \quad (18)$$

Substituting (18) into (17) and performing some algebraic manipulation gives

$$\begin{aligned} C(z,m) &= z^{-1} \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} g(kT+mT-jT) e(jT) z^{-k} \\ &= \sum_{j=0}^{\infty} e(jT) z^{-j} z^{-1} \sum_{(k-j)=0}^{\infty} g([k-j]T+mT) z^{-(k-j)} \\ &= \sum_{j=0}^{\infty} e(jT) z^{-j} G(z,m) = E(z) G(z,m) \end{aligned} \quad (19)$$

where $G(z,m)$ is the modified z-transform of $g(t)$. Equ. (19) is the modified z-transform counterpart corresponding to (6) of the regular z-transform relation. The relation (19) is the fundamental theoretical background to make it possible to establish the z-transform equation to relate the sampling instants signals (sampled signals) to the values of a continuous signal at time being skewed by mT from every sampling instant.

By the definition of the modified z-transform given in (17), $G(z,m)$ of $G(s)$ in Fig. 1 is evaluated as

$$G(z,m) = Z_m\left[\frac{1}{s+a}\right] = Z_m[e^{-at}] = z^{-1} \sum_{k=0}^{\infty} e^{-a(k+m)T} z^{-k} = \frac{e^{-amT}}{z-e^{-aT}} \quad (20)$$

Substituting (7) and (20) into (19) gives

$$C(z,m) = G(z,m) E(z) = \frac{e^{-amT} z}{(z-1)(z-e^{-aT})} \quad (21)$$

The case $m=1$ in the modified z-transform means that the signal is not delayed, which accordingly should imply that $C(z,m)=C(z)$. However, substituting $m=1$ into (17) gives

$$C(z,m)|_{m=1} = C(z,1) = z^{-1} \sum_{k=0}^{\infty} c([k+1]T) z^{-k} = C(z) - c(0) \quad (22)$$

which is an established property to relate the modified z-transform to the regular z-transform [1-3]. In this paper, however, it will be shown that there is a serious analytic misinterpretation of the definition of the modified z-transform in deriving (22). An example which invalidates (22) will be given first, and an investigation and a correction will be made analytically for (22).

Consider the system in Fig. 1 again. Simple setting $m=1$ in (21) gives

$$C(z,1)=C(z,m)|_{m=1}=\frac{e^{-aT}z}{(z-1)(z-e^{-aT})} \tag{23}$$

Now it can be immediately noticed that no matter whichever $c(0)$ between $c(0^+)=1$ and $c(0^-)=0$ is used, $C(z)$ given in (8) and $C(z,1)$ of (23) do not validate (22). The serious problem is that although the relation (22) has been established without any doubt in the well-cited references[1-4], the theory is not confirmed even by a typical example.

The discrepancy demonstrated with the above example should be investigated analytically. The parameter m in (16) or (17) for defining the modified z-transform can never be greater than or equal to 1. If $m\geq 1$ then the z transfer relation $C(z,m)=G(z,m)E(z)$ established in (19) becomes invalid since the convolution summation will include some extra terms. If it is the case, the modified z-transform is totally useless. Therefore it should be emphasized that the proper range of m for validating (19) is $0\leq m<1$.

By the insight given in the previous paragraph, the condition $m=1$ used in (22) should be understood as $m=1^-$, with which (22) is now revised as follows:

$$\begin{aligned} C(z,m)|_{m=1^-} &= C(z,1^-)=z^{-1}\sum_{k=0}^{\infty}c([k+1]T^-)z^{-k} \\ &= C^-(z)-c(0^-)=C^-(z) \end{aligned} \tag{24}$$

where $c(0^-)=0$ was used. Now comparing (15) with (23) confirms the validity of (24). Note that $c(0^+)$ and $c(0^-)$ are not involved at all in the final result of (24). Note also that when $c(0^+)=0$, which means that $c(t)$ does not have jump discontinuities at the sampling instants, all three of the right, left, and regular z-transforms become equal and therefore (22) is still valid for this limited situation, and in either of the case, (24) is a general formula.

By the similar approach given above, all the modified z-transform theorems are revised and/or clarified:

Multiplying both sides of (17) by z gives

$$zC(z,m)=\sum_{k=0}^{\infty}c(kT+mT)z^{-k} \tag{25}$$

and its inverse z-transform gives

$$c(kT+mT)=Z^{-1}[zC(z,m)] \tag{26}$$

which is $c(t)$ for the time duration $kT\leq t<kT+T$ for $k\geq 0$ and when m is varied between zero and one. Setting $m=0^+$ and $m=1^-$ in (26) gives, respectively,

$$c(kT^+)=Z^{-1}[zC(z,0^+)]\ ,\quad k\geq 0 \tag{27}$$

and

$$c([k+1]T^-)=Z^{-1}[zC(z,1^-)]\ ,\quad k\geq 0 \tag{28}$$

Setting $m=0^+$ in (25) gives

$$C^+(z)=C(z)=zC(z,0^+) \tag{29}$$

The inverse z-transform of (16) gives another expression of $c(t)$:

$$c(kT-T+mT)=Z^{-1}[C(z,m)] \tag{30}$$

where the time duration is $(k-1)T\leq t<kT$ for $k\geq 0$. Note that the time interval with $k=0$ for (30) is $-T\leq t<0$, for which $c(t)$ becomes zero. Setting $m=0^+$ and $m=1^-$ in (30) gives, respectively,

$$c([k-1]T^+)=Z^{-1}[C(z,0^+)]\ ,\quad k\geq 0 \tag{31}$$

and

$$c(kT^-)=Z^{-1}[C(z,1^-)]\ ,\quad k\geq 0 \tag{32}$$

and setting $m=1^-$ in (16) gives

$$C^-(z)=C(z,1^-) \tag{33}$$

which was given in (24), but repeated here for grouping the related formulas.

When a signal has jump discontinuities at the sampling instants, their sequence form is theoretically of interest. The z-transform of this sequence can be obtained from $C^+(z)$ and $C^-(z)$ given in (29) and (33), respectively:

$$D(z)=C^+(z)-C^-(z)=zC(z,0^+)-C(z,1^-) \tag{34}$$

and its inverse z-transform gives the actual jump sequence:

$$d_k=d(kT)=Z^{-1}[zC(z,0^+)-C(z,1^-)]\ ,\quad k\geq 0 \tag{35}$$

In summary, once the modified z-transform $C(z,m)$ is derived, then the z-transform $C(z)$, the time responses $c(t)$ for $kT\leq t<kT+T$, $c(kT^+)$, and $c(kT^-)$, and the jump discontinuity sequence $d(kT)$ can be obtained by (29), (26), (27), (32), and (35), respectively. For example, the output $c(t)$ of Fig. 1 between the sampling instants can now be obtained by taking the inverse z-transform after multiplying z to (21):

$$\begin{aligned} c(kT+mT) &= Z^{-1}\left[\frac{e^{-amT}z^2}{(z-e^{-aT})(z-1)}\right] \\ &= \frac{1-e^{-a(k+1)T}}{1-e^{-aT}}e^{-amT} \end{aligned} \tag{36}$$

which is the same as (10) with the substitution of $m=t/T-k$.

IV. Conclusions

In this paper two z-transforms were defined for a continuous signal $c(t)$ which is the output of a continuous plant $G(s)$ which again is subject to the impulse input: the right z-transform $C^+(z)$ and the left z-transform $C^-(z)$. It can be be easily concluded that $C^-(z)$ should be the same as $C^+(z)$ in case that $c(t)$ does not have any jump discontinuity, which, however, is not always the case. It turns out that the textbook-definition based z-transform $C(z)$ is actually the right z-transform $C^+(z)$. Using the new definitions of the z-transform, some revisions were made on the theorems of the modified z-transform so that they can be used without any analytic problems in analyzing the continuous signals, even with jump discontiunities at the sampling instants, of sampled data systems.

V. References

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