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Geometrically Non-Linear Analysis for Shallow Arch using the 3-Dimensional Curved Beam

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Abstract

This paper presents a geometrically non-linear formulation for the general curved beam element based on assumed strain fields and Timoshenko's beam theory. This general curved beam element is formulated from constant strain fields. And this element, designed in a local curvilinear coordinate system, is transformed into a global cartesian system in order to analyze effectively the general curved beam structures located arbitrarly in space. Numerical examples are presented to show the accuracy and efficiency of the present formulation. The results obtained from the present formulation are compared with those available in the literature and analysis by ANSYS.

1. Introduction

The curved beam elements are used for analyzing various curved beam structures effectively by the finite element method in general. Since deformation behaviors are coupled in the curved beam, these curved beam elements show the locking phenomena that have a direct effect on the accuracy of solutions. In order to reduce the locking phenomena in a curved beam element, many studies have been performed by means of various techniques. These are typically the strain element technique[1-3], the selective or reduced integration method[4] and the modified isoparametric elements[5,6], etc.. However, these previous works have been confined to beams curved and bent only in the plane of curvature, so that no torsion is involved. The original work including out-of-plane deformation, based on independent strain functions, was done by sabir[7]. However, it has been formulated on the basis of Euler beam theory. A few works[8-10] has been done on a general curved beam element, including the effect of shear, which is arbitrarily located in space, showing the in-plane and out-of-plane deformations as well as torsional deformations. These elements have been formulated on the basis of modified isopametric shape function in order to reduce the locking phenomena. Recently, the general curved beam elements based on the assumption of the strain fields which is well known as a means of removing the locking phenomena and Timoshenko's beam theory was developed by Choi and Lim[11]. These elements included the axial, in-plane, out-of-plane, bending and torsion deformations.

In the present work, geometrically nonlinear formulation for the general curved beam developed by Choi and Lim was derived. Numerical examples are presented to show the accuracy and efficiency of the present formulation. The results obtained from the present formulation are compared with those available in the literature and analysis by ANSYS.

2. General curved beam element

In a typical curved beam element, the local curvilinear coordinates x,y and z are defined as shown in Figure 1, and u, v, w, and θ_x, θ_y, θ_z are displacement and rotational components in the positive directions of each coordinate axis established at an arbitrary point on the beam axis. The strain-displacement relation on the basis of Timoshenko's beam theory are obtained as follows:

$$
\begin{aligned}
\varepsilon_{xx} &= \frac{du}{dx} + \frac{w}{R} \\
\chi_{xy} &= \frac{d\theta_y}{dx} \\
\gamma_{xz} &= \frac{u}{R} - \frac{dw}{dx} + \theta_y \\
\chi_{xz} &= \frac{d\theta_z}{dx} - \frac{\theta_x}{R} \\
\chi_{xx} &= \frac{d\theta_x}{dx} + \frac{\theta_z}{R} \\
\gamma_{xy} &= \frac{dv}{dx} - \theta_z
\end{aligned}
\tag{1}
$$

where R is the radius of curvature of the curved beam axis in the xz plane. In equation (1),

$$\{\bar{\varepsilon}\} = [\varepsilon_{xx}\ \chi_{xy}\ \gamma_{xz}\ \chi_{xz}\ \chi_{xx}\ \gamma_{xy}]^T \tag{2}$$

Suppose the strain vector $\{\bar{\varepsilon}\}$ has constant values in a discrete curved beam element such as

$$\{\bar{\varepsilon}\} = [C_1\ C_2\ C_3\ C_4\ C_5\ C_6]^T \tag{3}$$

where C_1, C_2, C_3, C_4, C_5 and C_6 are arbitrary constants. Equations (1) and (3) can be integrated to obtain general solutions of element displacement functions as follows:

$$\{q\} = [[\Phi_1][\Phi_2]]\{C\} = [\Phi]\{C\} \tag{4}$$

In equation (4),

$$\{q\} = [u\ v\ w\ \theta_x\ \theta_y\ \theta_z] \tag{5}$$

$$
[\Phi_1] = \begin{bmatrix}
0 & -R^2\phi & R & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & R^2\phi & R\phi \\
R & R^2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -R & 0 & 0 \\
0 & R\phi & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & R & 0
\end{bmatrix}
\tag{6}
$$

$$
[\Phi_2] = \begin{bmatrix}
0 & -\sin\phi & \cos\phi & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & \cos\phi & \sin\phi \\
0 & \cos\phi & \sin\phi & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\frac{1}{R}\cos\phi & -\frac{1}{R}\sin\phi \\
-\frac{1}{R} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\frac{1}{R}\sin\phi & \frac{1}{R}\cos\phi
\end{bmatrix}
\tag{7}
$$

{q} is the element displacement vector in the local curvilinear coordinate system. $[\Phi_1]$ and $[\Phi_2]$ represent the strain mode and the rigid body mode in a general curved beam element.

3. Derivation of linear and non-linear local stiffness matrix

The stiffness matrix can be derived from the total potential energy theorem. The total potential energy of a curved beam element is expressed as

$$\Pi^e = \frac{1}{2}\int_{x_i}^{x_e}\{\varepsilon\}^T[D]\{\varepsilon\}dx - \int_{x_i}^{x_e}\{q\}^T\{P\}dx \tag{8}$$

Generally, the strain vector is divided into linear and non-linear terms as follows

$$\{\varepsilon\} = \{\varepsilon_L\} + \{\varepsilon_N\} \tag{9}$$

The subscript L and N denote the linear and non-linear part, respectively.
$\{\varepsilon_L\}$ is a linear strain vector rearranged as

$$\{\varepsilon_L\} = [\varepsilon_{xx}\ \gamma_{xy}\ \gamma_{xz}\ \chi_{xx}\ \chi_{xy}\ \chi_{xz}]^T \tag{10}$$

which is related to the stress resltant vector $\{\sigma_L\}$. So, the linear stress-strain relations in general curved beam element may be expressed as

$$\{\sigma_L\} = [D_L]\{\varepsilon_L\} \tag{11}$$

where

$$[D_L] = \begin{bmatrix} EA & 0 & 0 & 0 & 0 & 0 \\ 0 & \beta^2 GA & 0 & 0 & 0 & 0 \\ 0 & 0 & \beta^2 GA & 0 & 0 & 0 \\ 0 & 0 & 0 & GI_x & 0 & 0 \\ 0 & 0 & 0 & 0 & GI_y & 0 \\ 0 & 0 & 0 & 0 & 0 & GI_z \end{bmatrix} \tag{12}$$

In equation (12), E is the Young's modulus, G is the shear modulus, β^2 is the shear correction factor, and A is the cross-sectional area. Thus, β^2GA is the shear rigidity, EI_y and EI_Z are the flexural rigidities in each direction, and GI_x is the torsional rigidity. $\{\varepsilon_N\}$ is nonlinear strain vector as

$$\{\varepsilon_N\} = [\varepsilon_n\ 0\ 0\ 0\ 0\ 0]^T$$
$$= [\frac{1}{2}(\frac{du}{dx})^2 + \frac{1}{2}(\frac{dv}{dx})^2 + \frac{1}{2}(\frac{dw}{dx})^2\ 0\ 0\ 0\ 0\ 0] \tag{13}$$

The linear and non-linear strain vector may be expressed in a matrix form with the generalized displacement vector {C} by rearranging the strain vector of equation (9) as

$$\{\varepsilon_L\} = [B_L]\{C\} \tag{14}$$

$$\{\varepsilon_{xx}\ \varepsilon_{n1}'\ \varepsilon_{n2}'\ \varepsilon_{n3}'\} = [B_N]\{C\} \tag{15}$$

where $\varepsilon_{n1}'=du/dx$, $\varepsilon_{n2}'=dv/dx$ and $\varepsilon_{n3}'=dw/dx$.
The assumed strain function matrix $[B_L]$ and $[B_N]$ in equations (14) and (15) are given by

$$[B_L] = \begin{bmatrix} 1&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0&0&0&0 \end{bmatrix} \tag{16}$$

$$[B_N] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\sin\phi}{R} & \frac{\cos\phi}{R} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & R & 1 & 0 & 0 & 0 & 0 & -\frac{\sin\phi}{R} & \frac{\cos\phi}{R} \\ 0 & -R & 0 & 0 & 1 & 0 & 0 & -\frac{\cos\phi}{R} & -\frac{\sin\phi}{R} & 0 & 0 & 0 \end{bmatrix} \tag{17}$$

The generalized displacements in equation (4) can be related to the nodal displacements by using the element boundary condition. Thus, its inverse relationship gives the generalized displacement vector {C} as follows

$$\{C\}=[A]^{-1}\{\Delta_L\} \tag{18}$$

where $\{\Delta_L\}=[q_1\ q_2]^T$

$$[A]=[\Phi]_{1,2}$$

The subscripts mean the positional substitution of each element nodal points. By substituting equations (9), (10) and (13) into equation (8), and rearranging the equation (8) with equation (14) and (15), and taking the first variation of equation (8) with respect to $\{\Delta_L\}$, an element equilibrium equation in the local curvilinear coordinate system is obtained as follows:

$$[A^{-1}]^T\left(\int_{x_i}^{x_e}[B_L]^T[D_L][B_L]\,dx+\int_{x_i}^{x_e}[B_N]^T[D_N'][B_N]\,dx\right)[A^{-1}]\{\Delta_L\}$$

$$=[A^{-1}]^T\int_{x_i}^{x_e}[\Phi]^T\{P\} \tag{19}$$

In equation (19), $[D_N']$ is a symmetrical constitution matrix resulting from the non-symmetric constitution matrix $[D_N]$ as follows:

$$[D_N]=EA\begin{bmatrix} 0 & \frac{1}{2}\varepsilon_{\kappa 1}' & \frac{1}{2}\varepsilon_{\kappa 2}' & \frac{1}{2}\varepsilon_{\kappa 3}' \\ \varepsilon_{\kappa 1}' & \varepsilon_{\kappa} & 0 & 0 \\ \varepsilon_{\kappa 2}' & 0 & \varepsilon_{\kappa} & 0 \\ \varepsilon_{\kappa 3}' & 0 & 0 & \varepsilon_{\kappa} \end{bmatrix} \tag{20}$$

$$[D_N']=EA\begin{bmatrix} 0 & \frac{1}{2}\varepsilon_{\kappa 1}' & \frac{1}{2}\varepsilon_{\kappa 2}' & \frac{1}{2}\varepsilon_{\kappa 3}' \\ \frac{1}{2}\varepsilon_{\kappa 1}' & \frac{1}{2}\varepsilon_{\kappa}+\frac{1}{2}\varepsilon_{xx} & 0 & 0 \\ \frac{1}{2}\varepsilon_{\kappa 2}' & 0 & \frac{1}{2}\varepsilon_{\kappa}+\frac{1}{2}\varepsilon_{xx} & 0 \\ \frac{1}{2}\varepsilon_{\kappa 3}' & 0 & 0 & \frac{1}{2}\varepsilon_{\kappa}+\frac{1}{2}\varepsilon_{xx} \end{bmatrix} \tag{21}$$

Thus, we can get the element stiffness matrix in the curvilinear coordinate system as

$$[K]=[K_L]+[K_N] \tag{22}$$

where $[K_L]=[A^{-1}]^T\int_{x_i}^{x_e}[B_L]^T[D_L][B_L]\,dx\ [A^{-1}]$

$$[K_N]=[A^{-1}]^T\int_{x_i}^{x_e}[B_N]^T[D_N][B_N]\,dx\ [A^{-1}]$$

4. Coordinate Transformation of an Element Matrix Equation

The nodal displacements in the local curvilinear coordinate system can be expressed in the form of the global displacement vector $\{\Delta_G\}$ of an element.

$$\{\Delta_L\}=[T_2][T_1]\{\Delta_G\} \tag{23}$$

where $[T_1]$: Transformation matrix between global cartesian set(X, Y, Z) and local cartesian set(x′, y′, z′) 1

$[T_2]$:Transformation matrix between local cartesian set(x′, y′, z′) and local curvilinear set(x, y, z)

By substituting this equation (23) into equation (19) and multiplying the transpose of $[T_2][T_1]$ on resulting equation, an element equilibrium equation is obtained in the global cartesian coordinate system as follows

$$[K_G]\{\varDelta_G\} = \{F_G\} \tag{24}$$

where $[K_G]=[T_1]^T[T_2]^T[K][T_2][T_1]$

$$\{F_G\} = [T_1]^T[T_2]^T[A^{-1}]^T\int_{x_i}^{x_e}[\Phi]^T\{P\}\,dx$$

In this equation, $\{K_G\}$ is the global stiffness matrix and $\{F_G\}$ is the equivalent nodal force vector of an element.

5. Tangential stiffness matrix

The geometrically non-linear equation is solved by means of the modified Riks method[12]. The tangential stiffness matrix used in this work can be obtained by taking the second variation of equation (8). The constitution matrix becomes of symmetric form as follows

$$[D_N''] = EA\begin{bmatrix} 0 & \varepsilon_{n1}' & \varepsilon_{n2}' & \varepsilon_{n3}' \\ \varepsilon_{n1}' & 3\varepsilon_n+\varepsilon_{xx} & 0 & 0 \\ \varepsilon_{n2}' & 0 & 3\varepsilon_n+\varepsilon_{xx} & 0 \\ \varepsilon_{n3}' & 0 & 0 & 3\varepsilon_n+\varepsilon_{xx} \end{bmatrix} \tag{25}$$

The tangential matrix $[K_T]^e$ can be obtained by substituting equations (12) and (25) into equations (22)-(24). The equilibrium equation in the incremental form is expressed as follows

$$[K_{TG}]\{\delta\varDelta_G\} = \{\delta F_G\} \tag{26}$$

Equation (26) has to be used only for the iterative procedure, and this equation can give the incremental displacement components according to the incremental loads.

6. Numerical expriment for the non-linear analysis

Typical shallow curved beam shown in Figure 2 is taken as the computational model. The values of the material properties are chosen as Young's modulus $E=1.0\times10^{-7}$ N/cm^2, poisson's ratio $\nu=0.3$ and shear correction factor $\beta^2=0.85$

6-1 In-plane large deflection analysis of a shallow arch

The hinged circular arch with a single static in-plane load at apex in vertical direction was analyzed for buckling using the general curved beam element. One half of the arch was idealized using 2, 4, 6 and 8 equal curved beam elements. Figure 3 and 4 show the predicted load-deflection curves of the arch for R/L=2.5 and variation of La/H in case of R/L=2.5. La means the arch-length in Figure 4. This arch was also analyzed by Choi and Lim[3], who used CSCC(Constant Strain Constant Curvature) element which is the 2-dimensional curved element. These results are exactly the same as those analyzed with CSCC element.

6-2 Out-of plane large deflection analysis of a shallow arch

In order to demonstrate the accuracy for the geometrical non-linear formulation of the general curved beam in case of out-of-plane large deflection analysis, the results were compared with those analyzed by ANSYS. For this analysis, The model which has R=1000 cm and θ=2.866° was selected to describe the analysis model which is close to straight beam in order to compare the analysis results using the general curved element with those anaylzed by ANSYS because ANSYS has the 3 dimensional straight beam element(BEAM4). Figure 5 shows the predicted load-deflection curves of the analysis model which is close to straight beam. The model analyzed by ANSYS is discretized into 12 elements and the model with the general curved beam element is discretized into 2, 4, 6 and 8 element. The results analyzed with the general curved element are agree with those analyzed by ANSYS very well. The same model used for in-plane large deflection analysis was selected for out-of-plane large deflection analysis to demonstrate the efficiency for the geometrically non-linear formulation of the general curved beam in case of out-of-plane large deflection analysis, but a single static load at apex was applied in out-of-plane lateral direction. Figure 6, 7, 8, and 9 show the predicted load-deflection curves for the comparision between the results with the general curved element and those analyzed by ANSYS. The model with the general curved beam element is discretized into 8 element and the model with ANSYS is discretized into 12 elements. These results show that a difference between the results with the general curved beam element and those analyzed by ANSYS exists remarkably according to increase of the ratio La/H.

7. Conclusion

A geometrically non-linear formulation is presented for the general curved beam element based on the assumptions of strain fields on the Timoshenko's beam theory. This general curved beam element is formulated from constant strain fields. And this element, designed in a local curvilinear coordinate system, is transformed into a global cartesian system in order to analyze effectively in space.

Numerical examples are presented to show the accuracy and efficiency of the present formulation. The results obtained from the present formulation were compared with those available in the literature and analysis by ANSYS. Numerical results show that the present formulation give a same results reported in reference[3] for in-plane large deflection of the shallow arch and a excellent agreement with those analyzed by ANSYS for the out-of-plane large deflection of the analysis model which is close to straight beam. Also, numerical results for out-of-plane large deflection of the shallow arch show a remarkable difference from those analyzed by ANSYS according to the variation of the aspect ratio. So, this formulation can be used efficiently for the geometrically non-linear analysis of the curved beam structures.

REFERENCE

1. D.G. Ashwell, A.B. Sabir and T.M. Roberts, "Further studies in the application of curved finite elements to circular arches." Int. J. Mech. Sci. 13, pp. 507-517, 1971
2. D.G. Ashwell and R.H. Gallagher, Finite Elements for Thin Shells and Curved Members, London, 1976
3. J.K. Choi and J.K. Lim, "Simple curved shear beam elements", Comm. Numer. Mech. Engng., 9, pp. 659-669, 1993
4. H. Stolarski and T. Belytschko, "Membrane locking and reduced integration for curved elements", J. Appl. Mech. ASME 49, pp. 172-178, 1981

5. C.R. Babu and G. Prathap, "A linear thick curved beam element", Int. J. Meth. Engng. 23, pp. 1313-1328, 1986
6. G. Prathap and C.R. Babu, "An isoparametric quadratic thick curved beam element, Int. J. Meth. Engng. 23, pp. 11583-1600, 1986
7. A.B. Sabir, Stiffness matrices for the general deformation(out-of-plane and inplane) of curved beam members based on independent strain functions. In The Mathematics of finite Elements and Application 2-Proceedings, Academic Press, london, pp. 411-421, 1976
8. Suman Dasgupta and Dipak Sengupta, "Horizontally curved isoparametric beam element with or without elastic foundation including effect of shear deformation", Comput. Struct., 29(6), pp. 967-973, 1988
9. B.P. Naganarayama and G. Prathap, "Consistency aspects of out-of-plane bending, torsion and shear in a quadratic curved element", Int. J. Meth. Engng. 30, pp. 431-443, 1990
10. G. Prathap and B.P. Naganarayama, "Analysis of locking and stress ocillations in a general curved beam element", Int. J. Meth. Engng. 30, pp. 177-200, 1990
11. J.K. Choi and J.K. Lim, " General curved beam elements based on the assumed strain fields", Comput. Struct., 55(3), pp. 379-386, 1995
12. M.A. Crisfield, "A fast incremental/iterative solution procedure that handles snap-through", Comput. Struct., 13, pp. 55-62, 1981
13. A.B. Sabir and A.C. Lock, "Large deflection, geometrically non-linear finite element analysis of circular arches", Int. J. Mech. Sci., 15, pp. 37-47, 1973

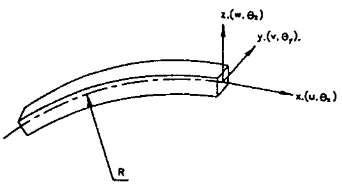

Figure 1 Geometry of a general curved beam element

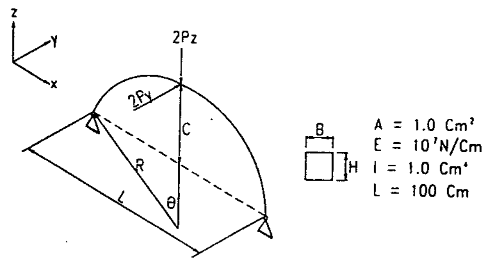

Figure 2 Shallow arch

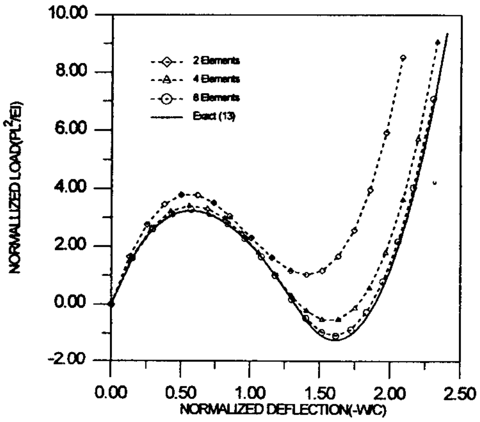

Figure 3 Load/Deflection Curves of the shallow arch for R/L=2.5(in-plane load)

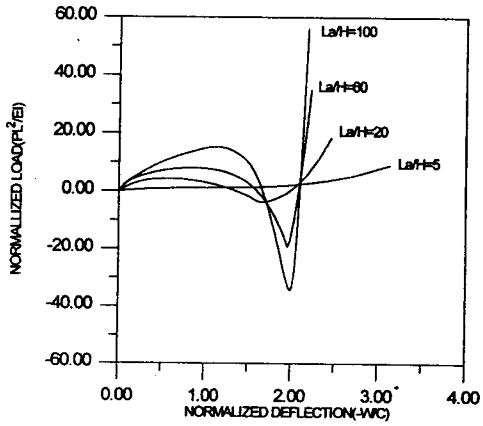

Figure 4 Load/Deflection Curves of the shallow arch for variations of La/H of R/L=2.5(in-plane Load)

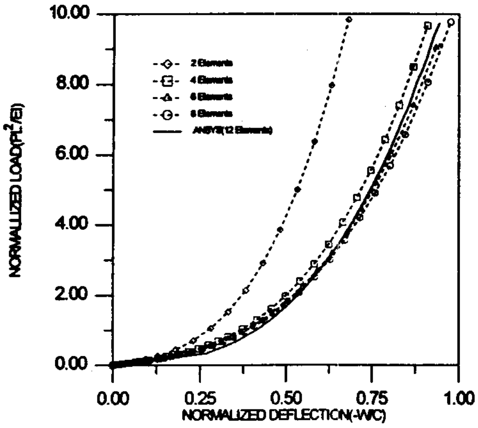

Figure 5 Load/Deflection Curves of the analysis model close to straight beam for the comparsion between Curved Beam and ANSYS result(out-of-plane load)

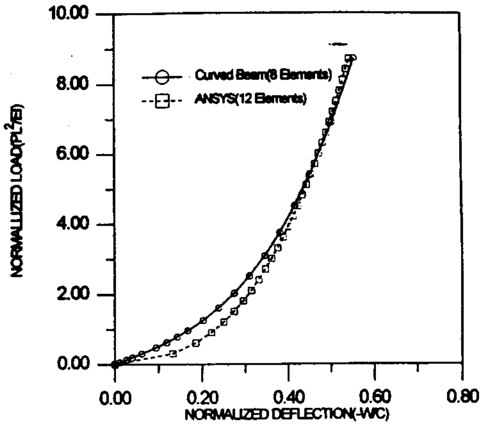

Figure 6 Load/Deflection Curves of the shallow arch for La/H=5(out-of-plane load)

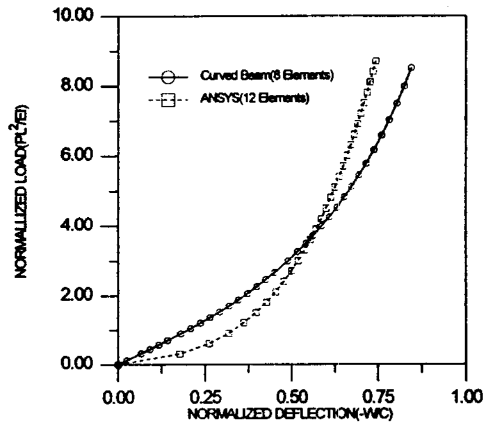

Figure 7 Load/Deflection Curves of the shallow arch for La/H=20(out-of-plane load)

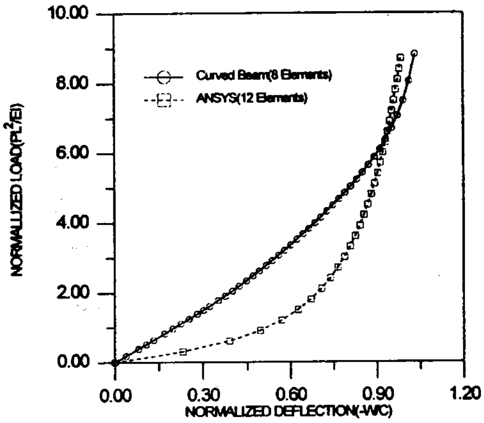

Figure 8 Load/Deflection Curves of the shallow arch for La/H=60(out-of-plane load)

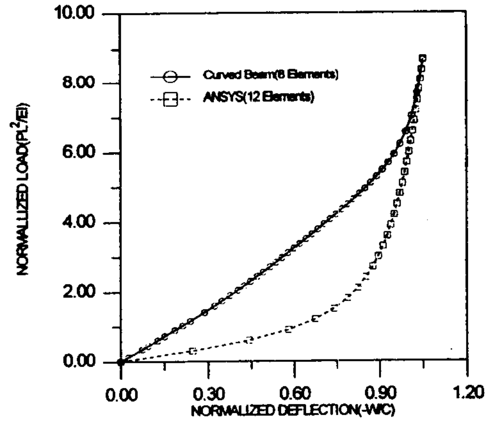

Figure 9 Load/Deflection Curves of shallow arch for La/H=100(out-of-plane)