막 구조물의 유한요소해석

Finite element analysis of wrinkling membranes

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ABSTRACT

A new iterative scheme is proposed for finite element analysis of wrinkling or tension structures. This enables us to update the stress state and the internal forces correctly taking into account the existence of wrinkling. The finite element implementation of the scheme is straightforward and simple, and only minor modifications of the existing total Lagrangian finite element codes for membranes are needed. The validity of the scheme is demonstrated via numerical examples for the torsion of a membrane and the quasi-static inflation of an automotive airbag, both made of isotropic or anisotropic elastic membranes.

1 Introduction

Analysis of wrinkling or tension structures, such as flexible membranes or fabric structures, has attracted substantial attention because of their increasing application in marine, space and terrestrial technology, and more specifically because of the simulation of airbags as a protection mechanism for drivers and passengers in automotive industry. There have been many works, theoretical and numerical, on the analysis of such wrinkling structures, for examples: Wagner[1] (1929), Reissner[2] (1938), Wu et al. [3] (1981), Roddeman et al. [4] (1987) and Steigmann et al. [5] (1989).

In this work, we introduce another scheme for the wrinkling analysis that can be used in the finite element analysis of anisotropic membranes and isotropic membranes. The scheme is based upon the observation that a local region of wrinkling is in the state of the uniaxial tension, and that the orientation and the magnitude of this uniaxial tension can be obtained from an invariant relationship between the normal strain component in the direction of the local uniaxial tension and the shear strain component in the presence of wrinkling. The scheme enables us to determine the wrinkling orientation in a straightforward manner and to reconstruct the stress state properly for wrinkled regions, so that the correct internal forces may be evaluated. We implement this scheme into a geometrically nonlinear finite element analysis using the total Lagrangian formulation. The finite element implementation of the scheme is very simple. We do not need any special finite elements, but

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only minor modifications of the existing total Lagrangian finite element codes for membranes are needed. We demonstrate the validity of the proposed scheme through numerical examples for an isotropic and for an orthotropic material, such as, the torsion of a membrane and the quasi-static inflation of circular airbags.

2 Basic equations and wrinkling analysis

Models describing the mechanical behavior of wrinkling membranes are usually based on the assumption that membranes have zero flexural stiffness. For the analysis of membranes with wrinkled regions, it is necessary to have some fundamental assumptions as follows. i) The configuration of the wrinkled region is controlled by negligibly small bending stiffness of the membrane. The exact shape of the membrane after wrinkling is not definable with only membrane theory. To describe the average membrane deformation that would be obtained after the wrinkles have been removed from the mid-plane, we define the fictitious non-wrinkled membrane which has the smooth surface as shown in Fig. 1. This fictitious non-wrinkled membrane gives only the average deformation. ii) Because the membrane is not able to support any compressive stresses, the membrane will wrinkle at once when a negative stress is about to appear. iii) The membrane is in the state of plane-stress.

In a small material element which is under locally homogeneous deformation in the presence of wrinkling, the stress is locally in the state of the uniaxial tension, and in the deformed configuration the direction of the uniaxial tension is perpendicular to the wrinkling direction. To describe deformations of a membrane, we rely upon the Cartesian coordinate systems as shown in Fig. 1. Let (X_1, X_2, X_3) denote a Cartesian coordinate of a material point in the undeformed configuration κ_o, and (x_1, x_2, x_3) a Cartesian coordinate of a material point in the deformed configuration $\kappa(t)$. For dealing with wrinkling, we take a local frame $(\hat{X}_1, \hat{X}_2)$ in κ_o such that the orientation of the $\hat{X}_1$-axis is lined up with the material line element of κ_o that is to be along the uniaxial tensile direction in the presence of wrinkling in $\kappa(t)$. For dealing with wrinkling, we take a local frame $(\hat{X}_1, \hat{X}_2)$ in κ_o such that the orientation of the $\hat{X}_1$-axis is lined up with the material line element of κ_o that is to be along the uniaxial tensile direction in the presence of wrinkling in $\kappa(t)$.

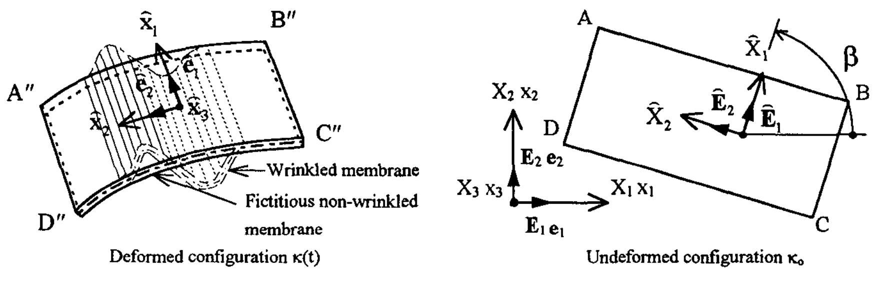


Fig. 1 The fictitious non-wrinkled membrane and coordinate systems.

Then the orientation of the X_1-axis is lined up with the material line element of κ_o that is normal to the orientation of the $\hat{X}_1$-axis. Moreover, we choose a local frame ($\hat{x}_1$, $\hat{x}_2$) defined on the fictitious non-wrinkled membrane in $\kappa(t)$ such that the $\hat{x}_1$-axis is along the uniaxial tension direction in the presence of wrinkling and the $\hat{x}_2$-axis is then aligned with the wrinkling direction. Note that a material line element along the $\hat{X}_1$-axis in κ_o is aligned with the $\hat{x}_1$-axis in $\kappa(t)$. However, a material line element along the $\hat{X}_1$-axis in κ_o is not mapped to be aligned with the $\hat{x}_2$-axis, which is along the wrinkling direction in $\kappa(t)$, unless the shear strain with respect to the ($\hat{X}_1$, $\hat{X}_2$) frame vanishes. Let $\mathbf{E}_I$, $\hat{\mathbf{E}}_I$, $\mathbf{e}_i$ and $\hat{\mathbf{e}}_i$ denote the unit base vectors along the coordinate axes X_I, $\hat{X}_I$, x_i and $\hat{x}_i$, respectively. Assuming that the Green strain $\mathbf{E}$ is small that the 2-nd Piola-Kirchhoff stress $\mathbf{S}$ may be approximated by the linear relationship with the Green strain, we can write the stress-strain relations referred to the X_1-X_2 coordinate system as $S^{IJ}=C^{IJKL}E_{KL}$, or in the "collapsed representation,"

$$\begin{Bmatrix} S^{11} \\ S^{22} \\ S^{12} \end{Bmatrix} = \begin{bmatrix} C^{11} & C^{12} & C^{13} \\ C^{21} & C^{22} & C^{23} \\ C^{31} & C^{32} & C^{33} \end{bmatrix} \cdot \begin{Bmatrix} E_{11} \\ E_{22} \\ 2E_{12} \end{Bmatrix} \tag{1}$$

where C^{IJKL} is the fourth order stiffness tensor and C^{IJ} is the component of a local "equivalent elasticity" matrix resulting from C^{IJKL}.

2.1 The state of stress and strain in wrinkling

Hereafter we will call the state of the uniaxial tension in the absence of wrinkling, "the state of the natural uniaxial tension," which is to be distinguished from the state of the uniaxial tension possibly with wrinkling, which is to be the genuine final state of deformation in the presence of wrinkling.

Note that the directions of the $\hat{X}_1$-axis and the $\hat{x}_1$-axis, which are the uniaxial tension direction in κ_o and $\kappa(t)$, respectively, are unknown and dependent upon a material point (X_1, X_2). The stress-strain relation referred to the $\hat{X}_I$ frame may be written as

$$\begin{Bmatrix} \hat{S}^{11} \\ \hat{S}^{22} \\ \hat{S}^{12} \end{Bmatrix} = \begin{bmatrix} \hat{C}^{11} & \hat{C}^{12} & \hat{C}^{13} \\ \hat{C}^{21} & \hat{C}^{22} & \hat{C}^{23} \\ \hat{C}^{31} & \hat{C}^{32} & \hat{C}^{33} \end{bmatrix} \cdot \begin{Bmatrix} \hat{E}_{11} \\ \hat{E}_{22} \\ 2\hat{E}_{12} \end{Bmatrix} \tag{2}$$

where $\hat{C}^{IJ} = C^{KL}T^{IK}T^{JL}$ with

$$[T] = \begin{bmatrix} \cos^2\beta & \sin^2\beta & 2\sin\beta\cos\beta \\ \sin^2\beta & \cos^2\beta & -2\sin\beta\cos\beta \\ -\sin\beta\cos\beta & \sin\beta\cos\beta & \cos^2\beta-\sin^2\beta \end{bmatrix} \tag{3}$$

We can eliminate strains $\hat{E}_{22}$ and $\hat{E}_{12}$ by using these uniaxial tension conditions, $\widehat{S}^{22} = \widehat{S}^{12} = 0$. Then the uniaxial stress-strain relation can be obtained as

$$\hat{S}^{11} = a \cdot \hat{E}_{11} \quad \text{with} \quad a = \frac{1}{\hat{C}^{23}\hat{C}^{32} - \hat{C}^{22}\hat{C}^{33}} \{ \hat{C}^{11}\left(\hat{C}^{23}\hat{C}^{32} - \hat{C}^{22}\hat{C}^{33}\right) + \hat{C}^{12}\left(\hat{C}^{21}\hat{C}^{33} - \hat{C}^{23}\hat{C}^{31}\right) + \hat{C}^{13}\left(\hat{C}^{31}\hat{C}^{22} - \hat{C}^{21}\hat{C}^{32}\right) \} \tag{4}$$

Once material data are set and the directions of the $\hat{X}_1$ - $\hat{X}_2$ axes are known, we can calculate the uniaxial stress at the natural uniaxial tension. Moreover, the strain components $\hat{E}_{22}$ and $\hat{E}_{12}$, satisfying the uniaxial tension condition *under the natural uniaxial tension* can be obtained as

$$\hat{E}_{22} = \frac{\hat{C}^{21}\hat{C}^{33} - \hat{C}^{23}\hat{C}^{31}}{\hat{C}^{23}\hat{C}^{32} - \hat{C}^{22}\hat{C}^{33}} \hat{E}_{11}, \text{ and } 2\hat{E}_{12} = \frac{\hat{C}^{22}\hat{C}^{31} - \hat{C}^{21}\hat{C}^{32}}{\hat{C}^{23}\hat{C}^{32} - \hat{C}^{22}\hat{C}^{33}} \hat{E}_{11} \tag{5.a,b}$$

During the pure wrinkling process, from the state of the natural uniaxial tension in the absence of wrinkling to the state of the uniaxial tension possibly with wrinkling, there is no change of the deformed coordinate $\hat{x}_1$ of a material point, while there is some change in the deformed coordinate $\hat{x}_2$. From these observations, it follows that during the pure wrinkling process there are no changes of the strain components $\hat{E}_{11}$ and $\hat{E}_{12}$, referred to the local Cartesian frame ($\hat{X}_1$, $\hat{X}_2$) in κ_o. This simple observation turns out to provide a useful clue for finding the wrinkling orientation, which is to be obtained as the direction of the uniaxial tension, we can devise an efficient scheme for searching for the wrinkling orientation, as will be shown in Section 2.2.

2.2 Wrinkling orientation

As discussed earlier, the wrinkled state is nothing but the state of the uniaxial tension. Hence the wrinkling orientation is determined by the direction of the uniaxial tension or equivalently by the orientation of the $\hat{X}_1$ - axis for a given deformation, which is given by β (see Fig. 1). To insure the uniaxial tension state, we use the following procedure:

i) check $\hat{E}_{11} > 0$ for an assumed value of $\hat{\beta}$

ii) set $\widehat{E}_{11} = \hat{E}_{11}$,

iii) calculate $\widehat{E}_{12}$ and $\widehat{E}_{22}$ from equation (5.a, b) (6)

iv) take $\beta = \hat{\beta}$ if $\widehat{E}_{12} = \hat{E}_{12}$ and $\widehat{E}_{22} \geq \hat{E}_{22}$.

2.3 Wrinkling criterion

From given strains and appropriate constitutive equations, whether the state of the membrane is taut, wrinkled or slack is determined based on wrinkling criteria. We will consider the wrinkling criterion based upon the principal stresses and strains by Roddeman et al.[4]. For an isotropic or an anisotropic material, the wrinkling criterion based upon the principal stresses and strains can be written as follows: let $S^1 \geq S^2$ and $E_1 \geq E_2$ as before. Then

i) If $S^2 > 0$, wrinkling does not occur. (taut)
ii) If $E_1 \leq 0$, biaxial wrinkling occurs. (slack) (7)
iii) Otherwise ($S^2 \leq 0$ and $E_1 > 0$), uniaxial wrinkling occurs. (wrinkled)

3 Finite Element Formulation

A total Lagrangian finite element formulation based upon the principle of virtual work is used for membrane finite element analysis (Bathe[6], 1982), into which the foregoing scheme of the wrinkling criterion and the search for the wrinkling orientation are incorporated. We may then obtain the following secant equation:

$$ {}^{(n+1)}F_I(c) = {}^{(n+1)}P_I \tag{8} $$

where **c** indicates the nodal displacement vector in the global finite element equation. Equation (8) represents the balance of the internal force and the external force for each nodal degree of freedom, and it is nonlinear in the nodal displacement ${}^{(n+1)}c$. For solution of this nonlinear equation, we rely upon a Newton type iterative scheme via Taylor series expansion, and we can finally obtain:

$$ {}^{(n,k)}K_{IJ}\,{}^{(n,k+1)}\Delta c_J = {}^{(n+1)}P_I - {}^{(n,k)}F_I \tag{9} $$

where ${}^{(n,k+1)}\Delta c_J$ is the nodal displacement increment for the (k+1)-th iteration, such that

$$ {}^{(n,k+1)}c_J = {}^{(n,k)}c_J + {}^{(n,k+1)}\Delta c_J \quad and \quad {}^{(n+1)}c_J = {}^{(n)}c_J + \sum_k {}^{(n,k)}\Delta c_J = \lim_{k\to\infty} {}^{(n,k)}c_J $$

$$ {}^{(n,k)}K_{IJ} = \frac{\partial\,{}^{(n,k)}F_I}{\partial c_J} = \int_{V^\circ} \frac{\partial N^\alpha}{\partial X_K} \frac{\partial\,{}^{(n,k)}x_m}{\partial X_L} C^{KLQP} \frac{\partial\,{}^{(n,k)}x_l}{\partial X_Q} \frac{\partial N^\beta}{\partial X_P} dV + \int_{V^\circ} \delta_{ml} \frac{\partial N^\alpha}{\partial X_K} {}^{(n,k)}S^{KL} \frac{\partial N^\beta}{\partial X_L} dV $$

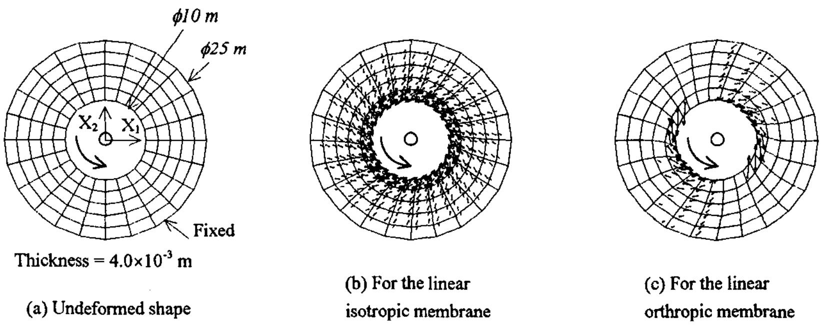
The iteration process for an equilibrium position is carried out in a two-stage procedure (Contri et al.[7], 1988) only for the first loading step. The first stage consists in searching for an equilibrium position of the membrane with both of the compressive stresses and the tensile stresses active. Once the equilibrium position is obtained, the compressive stresses are relaxed at the next stage as follows. Given a new estimation of the nodal displacements in the processes of equilibrium correction, at each integration point of an element the judgment is made on the wrinkling criterion, whether it is taut, wrinkled or slack. Here we use the wrinkling criterion (7) based upon the principal stresses and strains. After this decision, the following procedures will be used: In the taut situation, the stresses of the membrane are determined by the normal analysis without wrinkling. In the presence of wrinkling, the stresses are determined on the basis of the scheme for the wrinkled membrane, described in the previous section. In the slack situation, the stresses contain only zeros. When the membrane is in the wrinkled state, the special procedure discussed in Section 2.1 and 2.2 is required for the reconstruction of the stresses. For loading steps except the first, the iteration process for an equilibrium correction is carried out in the one-stage procedure which corresponds to the second stage of the first loading step.

4 Numerical examples

4.1 Torsion of a membrane

For the first example, we consider a circular membrane attached to a rigid disc at the inner edge and to a guard ring at the outer edge (Roddeman[8], 1991). Turning the rigid disc causes wrinkling of the membrane. The scheme which accounts for wrinkling should be used to calculate strains and stresses. For the finite element analysis, 120 four-node isoparametric membrane elements are used as shown in Fig. 2 (a). The nodal points on the outer circle are fixed in space. The nodal points on the inner circle are rotated over 10 degrees. A material behavior is assumed as follows: Young's modulus $E = 1.0\times10^5$ and Poission ratio $\nu = 0.3$ for a linear isotropic material; $E_{y1} = 1.0\times10^5$ Pa, $E_{y2} = 1.0\times10^6$ Pa, $\nu_{12} = 0.3$ and $G_{12} = 0.385\times10^5$ Pa, referred to the X_I frame in Fig. 2, for a linear orthotropic material.

The deformed shapes for the linear isotropic and orthotropic membranes are shown in Fig. 2 (b) and (c), respectively. Furthermore, the direction and the magnitude of the uniaxial tensile stress is indicated by using the direction and the length, respectively, of the arrow at each integration point in the wrinkled region. The magnitude of the wrinkling strain is also indicated by using circles of varying magnitude. Regions which are not indicated with arrows and the circles means a taut region. As expected, the isotropic problem shows itself to be rotationally symmetric. In the orthotropic membrane, wrinkling occurs mostly on the left-lower and the right-upper parts.



(a) Undeformed shape

(b) For the linear isotropic membrane

(c) For the linear orthropic membrane

Fig. 2 Deformed shapes, the uniaxial tensile stress and the wrinkling strain on a wrinkled region for torsion of a circular membrane (144 elements).

4.2 An inflatable circular airbag (Automotive airbag)

Now consider an inflatable circular airbag that initially consists of two flat circular pieces of fabric sewed together along the edge. In this case the unfilled (undeformed) structure exhibits a flat and stress-free surface which, when filled to a final volume, will experience stressing and wrinkling of the fabric. This wrinkling is due to the shrinkage in circumferential direction of the airbag as it is inflated. The inflatable circular airbag is modeled as two parallel circular planes using 3- and 4-node elements as shown in Fig. 3. In the initial configuration, the two circular planes of the front and the back coincide with each other. The action of the gas inside the airbag is assumed to be a uniform pressure distribution on the inner surfaces of the bag.

Consider the airbag to be made of the same two flat isotropic membranes. By applying appropriate boundary conditions in the horizontal mid-plane, we need only to model one quarter front of the bag. The finite element model for a quarter airbag consists of 20 total elements (4 in circumferential direction and 5 in radial direction). A linear isotropic material behavior is assumed as follows: $E=6.0\times10^7$ Pa and $\nu=0.3$. The thickness is 0.4×10^{-3} m. The airbag is subjected to a uniform pressure from 0 to 10 kPa. Fig. 4 shows the vertical displacement of the center point and the radial contraction of a point of the circumference with respect to the increase in the internal pressure for two cases: one obtained with wrinkling being taken into account and the other obtained from pure membrane theory with no wrinkling being taken into account. The difference between both cases is greatest in the low pressure region. As the pressure increases, the displacement difference is smaller since the wrinkled region decreases.

Consider next an airbag with front and back anisotropic membranes, modeled with 264 elements. The airbag is subjected to the uniform pressure of 5 kPa. The linear orthotropic material behavior is assumed as follows: $E_{y1} = 2.0\times10^8$ Pa, $E_{y2} = 2.0\times10^8$ Pa, $\nu_{12} = 0.1$, $G_{12} = 0.385\times10^5$ Pa : material principal angle $\theta_1 = 0°$ for the front plane membrane and material principal angle $\theta_2 = 45°$ for the back plane membrane (see Fig. 3).

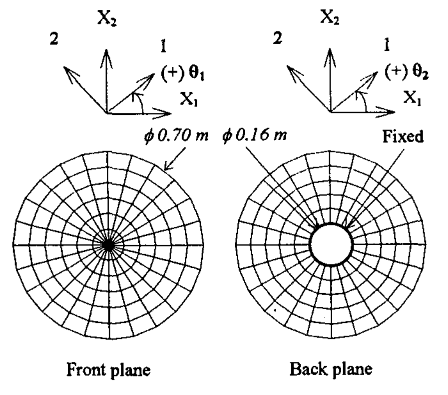
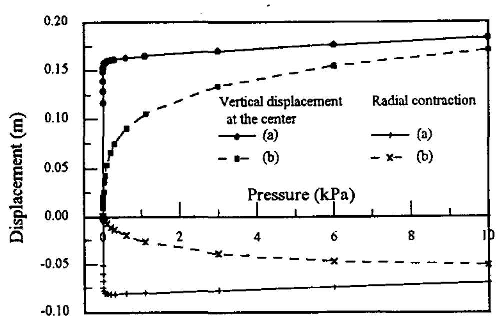


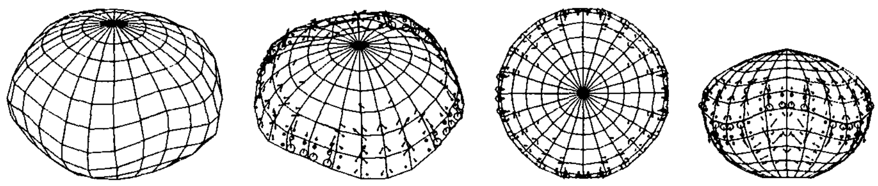
Fig. 3 The airbag model.



(a) : Displacements obtained with wrinkling being taken into account

(b) : Displacements obtained from pure membrane theory with no wrinkling being taken into

Fig. 4 The vertical displacement of the center point and the radial contraction of an inflatable circular airbag modeled with 20 elements (4×5) for a quarter plane.



(a) Deformed shape (b) Front plane (c) Top view (d) Side view

Fig. 5 Deformed shapes of an airbag (θ_1=0° and θ_2=45°).

The thickness is 0.4×10^{-3} m. Fig. 5 (a), (b), (c) and (d) show the overall deformed shape, the deformed shape of the front plane, the top view and the side view, respectively. Furthermore, the uniaxial tensile stress and the wrinkling strain are indicated, as in the aforementioned torsion case, at each integration point on the wrinkled region. The region with no arrows and circles means a taut region.

5 Conclusions

With the aid of the correct stress update based upon the observation regarding the invariant relation between some of the strain components referred to a coordinate system aligned with wrinkling, a simple but efficient scheme is proposed for finite element analysis of wrinkling. This scheme is found to be applicable to an anisotropic membrane and an isotropic membrane. Moreover it requires no special finite element development, but only minor modifications of the existing total Lagrangian finite element codes for membranes are needed. Two numerical examples have been used to demonstrate the validity of the proposed scheme: one is the torsion of a membrane and the other the inflation of an airbag used in the automotive applications.

References

1. Wagner, H., "Flat Sheet Metal Girders With Very Thin Metal Web," *Z. Flugtechn. Motorluftschiffahrt*, Vol. 20, 1929, pp. 200-314.
2. Reissner E., "On Tension Field Theory," *Proc. 5Th Int. Congr. Appl. Mech.*, 1938, pp. 88-92.
3. Wu, C. H., and Canfield, T. R., "Wrinkling in Finite Plane-Stress Theory," *Q. Appl. Math.*, 1981, pp. 179-199.
4. Roddeman, D. G., Oomens, C. W. J., Janssen, J. D., and Drukker, J., "The Wrinkling of Thin Membranes: Part I-Theory," ASME *J. Appl. Mech.*, Vol. 54, 1987, pp. 884-887.
5. Steigmann, D. J., and Pipkin, A. C., "Wrinkling of Pressurized Membranes," ASME *J. Appl. Mech.*, Vol. 56, 1989, pp. 624-628.
6. Bathe, K.-J., 1982, "Finite Element Procedures in Engineering Analysis," *Prentice-Hall*, 1982.
7. Contri, P., and Schrefler, B. A., "A Geometrically Nonlinear Finite Element Analysis of Wrinkled Membrane Surfaces by a No-compression Material Model," *Communications in Appl. Num. Meth.*, Vol. 4, 1988, pp. 5-15.
8. Roddeman, D. G., "Finite-Element Analysis of Wrinkling Membrane," *Communications in Appl. Num. Meth.*, Vol. 7, 1991, pp. 299-307.