한국전산구조공학회
1996년도 봄학술발표회 논문집

구조체의 위상학적 최적화를 위한 비선형 프로그래밍
NLP Formulation for the Topological Structural Optimization

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Abstract

The focus of this study is on the problem of the design of structure of undetermined topology. This problem has been regarded as being the most challenging of structural optimization problems, because of the difficulty of allowing topology to change. Conventional approaches break down when element sizes approach to zero, due to stiffness matrix singularity. In this study, a novel nonlinear programming formulation of the topology problem is developed and examined. Its main feature is the ability to account for topology variation through zero element sizes. Stiffness matrix singularity is avoided by embedding the equilibrium equations as equality constraints in the optimization problem. Although the formulation is general, two dimensional plane elasticity examples are presented. The design problem is to find minimum weight of a plane structure of fixed geometry but variable topology, subject to constraints on stress and displacement. Variables are thicknesses of finite elements, and are permitted to assume zero sizes. The examples demonstrate that the formulation is effective for finding at least a locally minimal weight.

1. Introduction

The problem of determining the optimal topology of structures modeled by finite elements is addressed. The problem is defined as follows: given a structure with fixed nodal locations and a list of possible element incidences (the ground structure), and given upper and lower bounds on displacements and stresses arising from loading conditions, find the subset of elements, and corresponding sizes, which minimize some function of the design variables. The design problem then includes configurational as well as sizing decisions. Examples of design variables include bar cross-sectional area, plate thickness, and beam moment of inertia..

We present a nonlinear programming (NLP) formulation for the topology problems. Our development addresses weight minimization of (possibly) inhomogeneous plate structure subject to stress, displacement; however, the topological formulation for other structures discretized by finite elements and other constraints types is possible and follows a similar development. We assume the optimization problem is solved by projected Lagrangian techniques[4)], which require at least zero- (values of objective and constraints) and first- (objective gradient and constraint Jacobian) order information to construct a linearly-constrained subproblem, the solution of which determines a search direction.

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For example, the popular sequential quadratic programming (SQP) algorithm uses a quadratic programming subproblem to determine the search direction.

The obvious approach to solving topological optimization problems of allowing zero lower bounds on the size of elements breaks down with a conventional *hierarchical* formulation, i.e., a formulation which eliminates state variables (e.g. displacements, stress) from the model by solving the equilibrium equations at each optimization iteration. In this formulation, an analysis is performed to provide zero-order information for constraints, and sensitivity information is computed based on the analysis, yielding first-order information. The results are then used to construct a linearly-constrained subproblem, the solution of which is used to find a new search direction. Consequently, the number of optimization variables is equal to the number of design variables, and the constraints are limited to those dictating design. The resulting Jacobian and Hessian matrices are small and dense. If critical element sizes assume zero value (as desired in topological optimization), stiffness matrix singularity can ensue, and the algorithm terminates at a suboptimal solution. The simple fix of altering the structural model as an element size reaches a small value ε is not satisfactory: if ε is too large, the decision to alter the model by dropping an element may be premature (which is important since the element can not be recovered, since it is not contained within the model); if ε is too small, the resulting stiffness matrix may be ill-conditioned, leading to poor calculated displacements and stresses (which can mislead the optimization).

On the other hand, the *simultaneous* formulation includes the equilibrium equations as equality constraints, and requires only their *evaluation* and not their *solution* at each iteration. Its use results in a larger number of constraints and variables, which now include state variables as well as design variables as unknowns. Even though the Jacobian and Hessian matrices are larger, they are sparse, and the total number of nonzeroes is typically much smaller than in the hierarchical formulation. With proper exploitation of sparsity, and especially if the behavior is nonlinear, greater efficiency can be achieve. The optimization process now moves towards a set of variables which simultaneously satisfy equilibrium and minimize the objective. In contrast to the hierarchical formulation, invertibility of stiffness matrix is not required, and sub-structures can be created by deleting elements (which might cause singularity of the stiffness matrix of the original structure). This is consequence of the fact that only the residual of the equilibrium equations is required for zero-order information, and only the pseudo-force vectors associated with sensitivity analysis and an evaluation of the stiffness matrix are required for first order information. The linearly-constrained subproblem is well-posed, the Jacobian matrix has full row rank, and a numerical solution to the subproblem can be readily obtained.

2. NLP Formulation for the Optimal Topology Problem

2.1 General Nonlinearly Constrained Optimization

The general constrained optimization problem may be expressed as :

$$
\begin{array}{llll}
\textit{minimize} & F(\mathbf{x}) & & \text{objective function} \\
\textit{subject to:} & & & \\
 & g_i(\mathbf{x}) = 0 & i = 1, 2, \ldots, m_e & \text{equality constraints} \\
 & g_i(\mathbf{x}) \geq 0 & i = m_{e+1}, \ldots, m & \text{inequality constraints}
\end{array} \tag{2.1}
$$

where

m_e number of equality constraints

m number of total constraints

$\mathbf{x}$ a vector containing optimization design variables

The objective function or any of the constraints imposed on the variables do not always involve only linear function. Most often the case in design optimization involves nonlinear function. Then, the problem is said to be one of the class of nonlinear programming problems (NLP).

2.2 Hierarchical Method and Simultaneous Method

2.2.1 Hierarchical Method

Figure 2.1 shows the general optimum design process[1]. Thus conventional optimization formulation for structural design (the hierarchical method) does not include equilibrium equations (**Ku**=**P**) in its constraints.

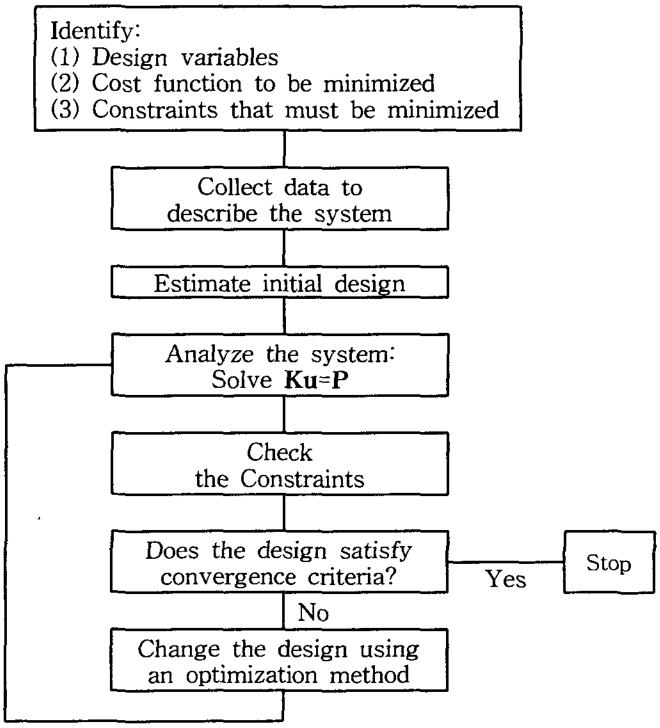

Figure 2.1 Optimum design process

It eliminates displacement variables in constraints by solving equilibrium equations at each optimization iteration. Hence, the hierarchical formulation is expressed as follows (assume the constraints are related to only displacements and stresses) :

$$g_1(\mathbf{x}) = C_1[\mathbf{K}(\mathbf{x})^{-1}\mathbf{P}(\mathbf{x})] - \mathbf{u}_b \geq 0 \qquad (2.2)$$

$$g_2(\mathbf{x}) = C_2[\mathbf{K}(\mathbf{x})^{-1}\mathbf{P}(\mathbf{x})] - \sigma_b \geq 0 \qquad (2.3)$$

where

$g_1(\mathbf{x})$	displacement constraints
$g_2(\mathbf{x})$	stress constraints
C_1, C_2	matrix of constant coefficients
$\mathbf{K}(\mathbf{x})$	stiffness matrix
$\mathbf{P}(\mathbf{x})$	a vector of applied loads

$\mathbf{u}_b$	displacement limits
σ_b	stress limits

Clearly, these constraints are not meaningful when element sizes assume zero values, since the stiffness matrix becomes singular and its inverse no longer exists.

2.2.2 Simultaneous Method

The simultaneous formulation directly includes the equilibrium equations as equality constraints. The simultaneous formulation is expressed as follows:

$$g_1(\mathbf{x}) = C_1\mathbf{u}(\mathbf{x}) - \mathbf{u}_b \geq 0 \tag{2.4}$$

$$g_2(\mathbf{x}) = C_2\mathbf{u}(\mathbf{x}) - \sigma_b \geq 0 \tag{2.5}$$

$$g_e(\mathbf{x}) = \mathbf{K}(\mathbf{x})\mathbf{u}(\mathbf{x}) - \mathbf{P}(\mathbf{x}) = 0 \tag{2.6}$$

The problem size becomes larger than that of the hierarchical method because of the larger number of variables in the constraints. But by including the equilibrium equations as equality constraints, one can avoid its singularity. It does not require stiffness matrix inversion. It requires only their evaluations, not their solution, at each optimization iteration.

2.3 NLP Formulation for the Topological Structural Optimization

As stated above, our development addresses the minimum weight of structures. It incorporates zero sizes; hence, the simultaneous method is used to insure that matrix singularity is avoided.

Formulation

The NLP for the optimal topology is stated as follows :

objective function:

$$minimize\ \mathrm{F} = \text{total weight} = \sum_{i=1}^{k} A_i \rho_i t_i \tag{2.7}$$

constraints:

subject to:

Equilibrium equation:

$$\mathbf{Ku} - \mathbf{P} = \mathbf{0} \tag{2.8}$$

Stress constraints:

$$\sigma^L_i \leq \sigma_i \leq \sigma^U_i \qquad i=1, \cdots k \tag{2.9}$$

Displacement constraints:

$$\mathbf{u}^L \leq \mathbf{u} \leq \mathbf{u}^U \tag{2.10}$$

Thickness constraints:

$$t^L_i \leq t_i \leq t^U_i \qquad i=1, \cdots k \tag{2.11}$$

Parameters are defined as:

k	number of total elements
n	number of degree of freedom after applying boundary condition
$\mathbf{K}$	$n \times n$-stiffness matrix
$\mathbf{P}$	n-vector of applied nodal loads
σ^L_i, σ^U_i	stress lower (upper) bounds of element i
$\mathbf{u}^L$, $\mathbf{u}^U$	n-vector of nodal displacement lower (upper) bounds
t^L_i, t^U_i	thickness lower (upper) bounds of element i
A_i	area of element i
ρ_i	density of element i

and the variables are defined as:

t_i thickness of element i

$\mathbf{u}$ n-vector of nodal displacement

Remarks

- All functions (2.7)-(2.11) are assumed to be continuously differentiable.
- The nonlinearity in this formulation is found in the equilibrium equations (2.8) and stress constraints (2.9), which include bilinear product of displacement and thickness. The objective function (2.7) and all other constraints are linear.
- If none of the t^L_i is zero, then the NLP (2.7)-(2.11) is no longer a topological design problem and topology is fixed by the thickness lower bounds.
- There is no guarantee that a unique minimum exist, or that a local minimizer coincides with a global minimizer.
- A single stress constraints (2.9) or displacement constraints (2.10) can be chosen, if needed.

2.4 Sequential Quadratic Programming Algorithm

The sequential quadratic programming (SQP) method is generally regarded as the best technique solving the NLP (2.1)[6], and will be the method of choice in this study. SQP can be derived as a Newton method for solving the first-order constrained stationary conditions[4]. It is based on the iterative formulation and solution of quadratic programming subproblems. These subproblems are defined by an objective function consisting of a quadratic approximation of the Lagrangian function, the minimization of which is subject to linear approximations of the original constraints. That is:

$$\textit{minimize} \quad \frac{1}{2}\mathbf{p}_k^T\mathbf{B}(\mathbf{x}_k, \lambda_k)\mathbf{p}_k + \nabla F(\mathbf{x}_k)^T\mathbf{p}_k$$

subject to:

$$\nabla g_i(\mathbf{x}_k)^T\mathbf{p}_k + g_i(\mathbf{x}_k) = 0 \qquad i = 1, 2, \ldots, m_e$$

$$\nabla g_i(\mathbf{x}_k)^T\mathbf{p}_k + g_i(\mathbf{x}_k) \geq 0 \qquad i = m_{e+1}, \ldots, m$$

$$\mathbf{x}^L - \mathbf{x}_k \leq \mathbf{p}_k \leq \mathbf{x}^U - \mathbf{x}_k$$

where $\mathbf{B}_k$ is a positive definite approximation of the Hessian of the Lagrangian function. $\mathbf{x}_k$ represents the current iterate points. Let $\mathbf{p}_k$ be the solution of the subproblem. A line search is used to find a new point $\mathbf{x}_{k+1}$, where

$$\mathbf{x}_{k+1}=\mathbf{x}_k + \alpha\mathbf{p}_k \qquad \alpha \in (0,1]$$

such that a merit function will have a lower function value at the new point. The augmented Lagrange function is used here as the merit function. When optimality is not achieved, $\mathbf{B}_k$ is updated according to the BFGS formula.

Remarks

- SQP applied to this problem requires at least the gradient of objective function and Jacobian matrix of the constraint set with respect to the optimization variables. Second derivative informations can be approximated from differences of first derivatives. These techniques are known as quasi-Newton method.

3. Examples of NLP for Topology Optimization

The NLP formulation of the topology problem is tested with 12×20 element rectangular model fixed at bottom depicted in Figure 3.1. This model contains 140 D.O.F. and 120 elements. This problem has one equality constraint set (equilibrium equations) and four inequality constraint sets (three stress and one thickness constraints). Hence, it has 260 (number of D.O.F.+number of total elements) variables and 620 constraints. Therefore, its

Jacobian size is 620×260. The initial guesses are computed using the equilibrium equations and an initial guess for thickness of 0.5*cm*. Exact derivatives are used to construct the gradient of objective function and Jacobian matrix of the constraints. Initial guesses for the state variables (displacements) are computed from the equilibrium equations for an initial design to initiate the SQP method.

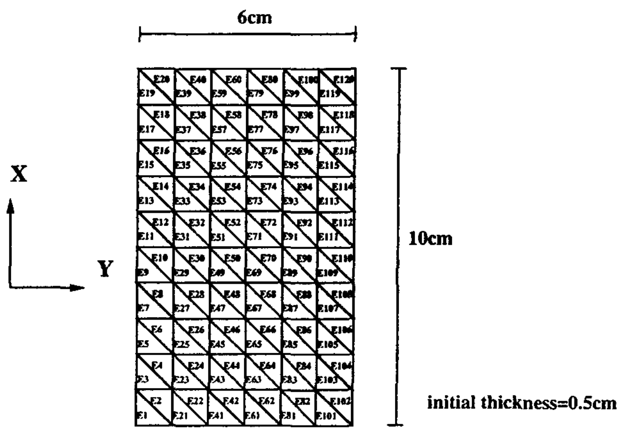

Figure 3.1 Model

Common data for problems

- Aluminum (Al 6061-T6) is the material, i.e.

  $E = 70Gpa$ $\qquad \sigma_y = 240Mpa$ $\qquad \tau_y = 140Mpa$

  $\rho = 0.002710kg/cm^3$ $\qquad \nu = 0.34615$
- Triangular finite elements are used.
- The structure is in plane stress.
- For stress constraints, 2 principle stress (σ_1, σ_2) and maximum shear stress (τ_{max}) are calculated for each element, and these stresses should be less than (or equal to) the maximum tension (compression, shear) stresses. That is,

$$\sigma_1 \leq \sigma_y$$
$$\sigma_2 \geq -\sigma_y$$
$$\tau_{max} \leq \tau_y$$

- For thickness constraints, following is used:

$$0. \leq t \leq 10^7$$

- Density and areas of all elements are equal in each example, hence, the objective function is set to $F = \Sigma t_i$.
- If thickness of any element reaches zero, stress in that element is defined as zero.
- SQP terminates when the optimality condition is less than 10^{-7}

Figures 3.2-3.4 show the optimal topology. Interestingly, Elements 18, 38, 58 and 119 in case 2.1 and Elements 85 and 15 in case 2.2 have nonzero thickness, but these elements have zero thickness in case 2.3 (which is the combination of case 2.1 and case 2.2). Symbolically,

Applied loads: $\mathbf{P}_{case2.1} + \mathbf{P}_{case2.2} = \mathbf{P}_{case2.3}$

however,

Resulting Topology: $\boldsymbol{T}_{case2.1} + \boldsymbol{T}_{case2.2} \neq \boldsymbol{T}_{case2.3}$

Number of iterations to converge to the optimality and minimum weights are shown in Table 3.1.

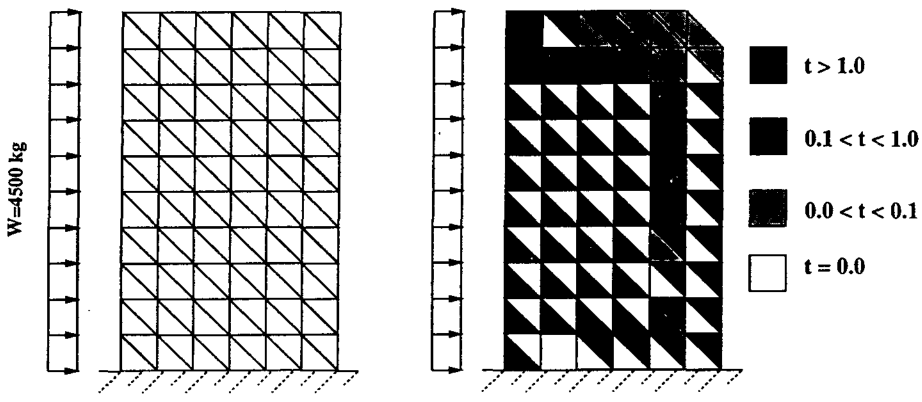

Figure 3.2 Optimal topology of case 2.1

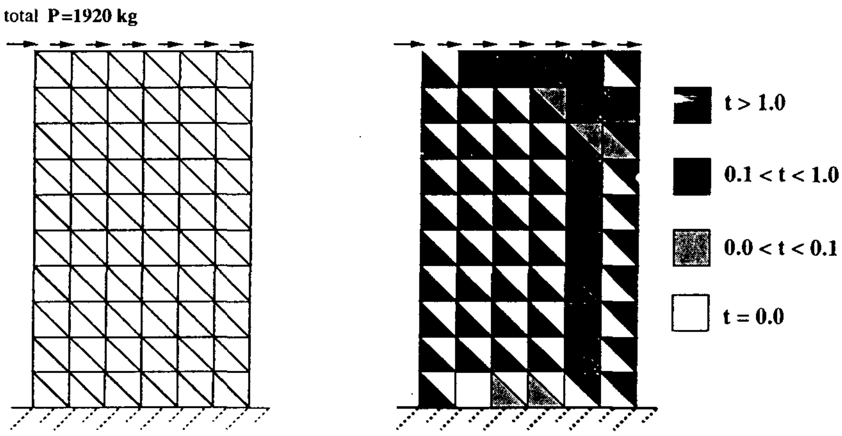

Figure 3.3 Optimal topology of case 2.2

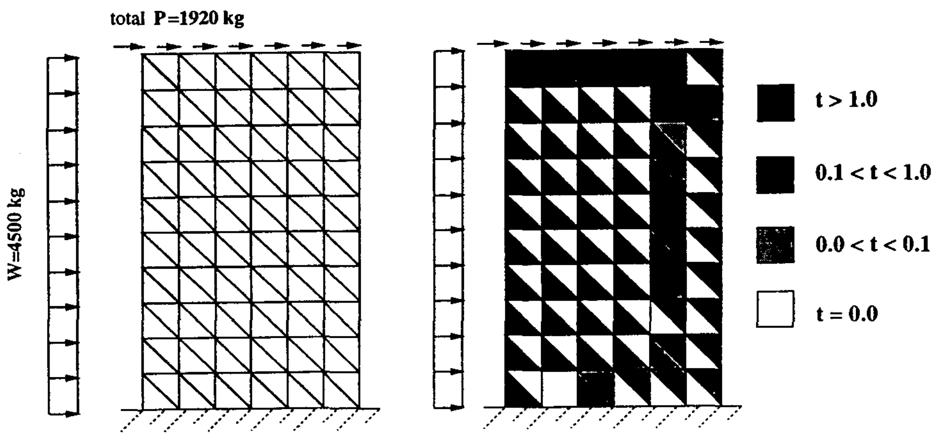

Figure 3.4 Optimal topology of case 2.3

Case number	Initial weight (kg)	No. of iteration	Optimal weight (kg)
Case .2.1	0.0813	40	0.02885943
Case 2.2	0.0813	24	0.02879123
Case 2.3	0.0813	50	0.05677134

Table 3.1 Optimal results

4. Conclusion

We have presented an NLP formulation for the optimal topology problem of structure. This problem has been regarded as posing the greatest difficulty to successful optimal design. The formulation guarantees at least a local minimum. Potential singularity of the stiffness matrix is avoided by embedding the behavioral equations as equality constraints in the optimization problem. Arbitrary objective functions, stress and displacement constraints, and upper and lower bounds on and linking of the design variables can be easily handled. The formulation is demonstrated on a number of examples of topology optimization of plate structures loaded in plane, and shown to be robust under a variety of constraints.

In this study, the formulation was tested under a single loading condition. However it would be desirable to apply the formulation to multiple loading conditions. Thus, we will test the formulation under multiple loading conditions in the not-too-distance future.

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