Multidimensional Linear Interpolation is a Spetial Form of Tsukamoto's Fuzzy Reasoning

*Kyong-Sik Om, Hee-Chan Kim†, Byoung-Goo Min†

*Dept. of Biomedical Engineering, College of Engineering, Seoul National University

†Dept. of Biomedical Engineering, College of Medicine, Seoul National University

Abstract

This paper examines the relationship between multidimensional linear interpolation (MDI) and fuzzy reasoning, and shows that an MDI is a special form of Tsukamoto's fuzzy reasoning. From this result, we find a new possibility of defuzzification scheme.

Index Terms

MDI : Multidimensional Linear Interpolation

STM system : Special Tsukamoto's Membership system

I. Introduction

Interpolation technique is used in the application of signal processing widely [1], and there is also a study of application to fuzzy learnig [2]. This papers examines the relationship between multidimensional linear interpolation (MDI) and fuzzy reasoning, and shows that an MDI is a special form of Tsukamoto's fuzzy reasoning [3]. From this result, we find a new possibility of defuzzification scheme.

I. Multidimensional Linear Interpolation

Before we proceed, it is necessary to comprehand that what we mean the MDI is the problem of interpolating on a mesh that is Cartesian, i.e., has not tabulated function values at "random" points in n-dimensional space rather than at the vertices of a rectangular array. For simplicity, we consider only the case of three dimensions, the cases of two and four or more dimensions being analogous in every way. If the input variable arrays are $x_{1a}[\ ]$, $x_{2a}[\ ]$, and $x_{3a}[\ ]$, the output y (x_1, x_2, x_3) has following relation [4].

$$y_a[m][n][r] = y(x_{1a}[m], x_{2a}[n], x_{3a}[r]) \quad (1)$$

The goal is to estimate, by interpolation, the function y at some untabulated point (x_1, x_2, x_3). If x_1, x_2, x_3 satisfy

$$\begin{cases} x_{1a}[\ m\ ] \le x_1 \le x_{1a}[\ m+1\ ] \\ x_{2a}[\ n\ ] \le x_2 \le x_{2a}[\ n+1\ ] \\ x_{3a}[\ r\ ] \le x_3 \le x_{3a}[\ r+1\ ], \end{cases} \quad (2)$$

the grid points are

$$
\begin{aligned}
y_1 &= y_a[\ m\ ][\ n\ ][\ r\ ] \\
y_2 &= y_a[\ m\ ][\ n\ ][\ r+1\ ] \\
y_3 &= y_a[\ m\ ][\ n+1\ ][\ r\ ] \\
y_4 &= y_a[\ m\ ][\ n+1\ ][\ r+1\ ] \\
y_5 &= y_a[\ m+1\ ][\ n\ ][\ r\ ] \\
y_6 &= y_a[\ m+1\ ][\ n\ ][\ r+1\ ] \\
y_7 &= y_a[\ m+1\ ][\ n+1\ ][\ r\ ] \\
y_8 &= y_a[\ m+1\ ][\ n+1\ ][\ r+1\ ].
\end{aligned} \quad (3)
$$

The final 3-dimensional linear interpolation is

$$
\begin{aligned}
y(\ x_1\ ,\ x_2\ ,\ x_3\ ) = \\
(1-u)(1-v)(1-w)y_1 \\
+ (1-u)(1-v)(w)y_2 \\
+ (1-u)(v)(1-w)y_3 \\
+ (1-u)(v)(w)y_4 \\
+ (u)(1-v)(1-w)y_5 \\
+ (u)(1-v)(w)y_6 \\
+ (u)(v)(1-w)y_7 \\
+ (u)(v)(w)y_8
\end{aligned} \quad (4)
$$

where

$$
\begin{aligned}
u &= \frac{x_1 - x_{1a}[m]}{x_{1a}[m+1] - x_{1a}[m]} \\
v &= \frac{x_2 - x_{2a}[n]}{x_{2a}[n+1] - x_{2a}[n]} \\
w &= \frac{x_3 - x_{3a}[r]}{x_{3a}[r+1] - x_{3a}[r]} .
\end{aligned} \quad (5)
$$

(u, v, and w each lie between 0 and 1.)

We can see the estimated y uses 2^n table terms if n-dimensions, and it satisfies 8 terms in case of three dimensions as above.

III. Tsukamoto's Fuzzy Reasoning

Tsukamoto used monotonic membership functions for linguistic terms [3]. As an example, consider the case of two input variables and one output variable.

$$
\begin{aligned}
R1 &: If\ x_1 = A_{11}\ \text{and}\ x_2 = A_{21,}\ \ then\ \ y_1 = B_{1.} \\
R2 &: If\ x_1 = A_{12}\ \text{and}\ x_2 = A_{22,}\ \ then\ \ y_2 = B_{2.} \\
R3 &: If\ x_1 = A_{13}\ \text{and}\ x_2 = A_{23,}\ \ then\ \ y_3 = B_{3.} \\
R4 &: If\ x_1 = A_{14}\ \text{and}\ x_2 = A_{24,}\ \ then\ \ y_4 = B_{4.}
\end{aligned} \quad (6)
$$

where

y_i : inferred variable of the consequence.

x_1, x_2 : variables of the premise.

A_{1i}, A_{2i} : normalized fuzzy sets over the input domain U and V.

B_i : normalized fuzzy sets over the output domain W.

If we define the fuzzified value A_1' and A_2' for input $x_1 = x_1^0$, $x_2 = x_2^0$, as fuzzy singletons as follows,

$$
\begin{aligned}
A_1' &= \begin{cases} 1, & \text{if } x_1 = x_1^0, \\ 0, & \text{otherwise}, \end{cases} \\
A_2' &= \begin{cases} 1, & \text{if } x_2 = x_2^0, \\ 0, & \text{otherwise}, \end{cases}
\end{aligned} \quad (7)
$$

the compatibility w_i for ith rule is,

$$ w_i = A_{1i}(x_1^0) \wedge A_{2i}(x_2^0), \quad (8) $$

or

$$ w_i = A_{1i}(x_1^0) A_{2i}(x_2^0). \quad (9) $$

Here, Eq. (8) means logical product ($\wedge$), and Eq. (9) algebraic product (•). The result y_i^* inferred from R_i is defined as follow :

$$ w_i = B(y_i^*) \rightarrow y_i^* = B^{-1}(w_i). \quad (10) $$

The infered value y^* from all rules usually calculated by *weighted* *combination method* as follows :

$$y^* = \frac{\sum_{i=1}^{4} w_i B_i^i(w_i)}{\sum_{i=1}^{i=4} w_i} \qquad (11)$$

B_i must be monotonic, whereas A_{1i}, and A_{2i} have no restriction of shape.

IV. Expression of Multidimensional Linear Interpolation from Fuzzy Reasoning

Triangular membership functions are used to subdivide the input universe. A fuzzy set A_i defined by triangular membership functions has the form

$$\mu(x) = \begin{cases} \dfrac{x - a_{i-1}}{a_i - a_{i-1}}, & \text{if } a_{i-1} \le x \le a_i \\ \dfrac{-x + a_{i+1}}{a_{i+1} - a_i}, & \text{if } a_i \le x \le a_{i+1} \\ 0, & \text{otherwise.} \end{cases} \qquad (12)$$

The point a_i will be referfed to as the midpoint of A_i. The leftmost and rightmost fuzzy regions are truncated with the midpoint as the leftmost and rightmost position, respectively. And the straight line membership functions are used for the output universe W. Fig. 1 and Fig. 2 help to understand the above relations.

From now, previous membership system will be called a STM (Special Tsukamoto's Membership) system. As an example, we consider three input variables. If we use fuzzy singletons as inputs x_1^0, x_2^0 and x_3^0, the compatibility w_i is $A_{1i}(x_1^0) \nabla A_{2i}(x_2^0) \nabla A_{3i}(x_3^0)$ where ∇ denotes triangular norms (T-norm). And $w_i = 0$ if $A_{1i}(x_1^0) = 0$ or $A_{2i}(x_2^0) = 0$ or $A_{3i}(x_3^0) = 0$. If the number of input variables are n, there are at most 2^n cases of possible maximum rules that have non-zero w_i, and the possible number of non-zero rules is eight for the case of $n = 3$. Possible rulse are smaller than 2^n when there are variables of premise which lies on midpoints of A_i. This fact coincides with the number of table terms which are used in MDI. If we use algebraic product for w_i,

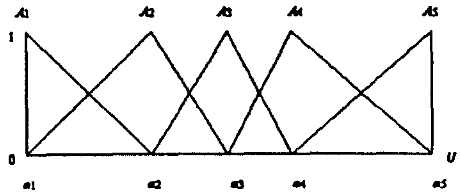

Fig. 1. Triangular decomposition of input domain.

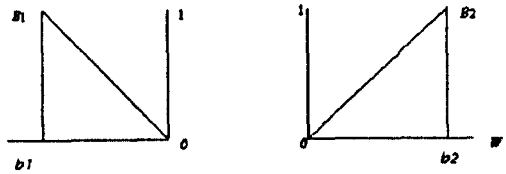

Fig. 2. Representing output domain using straight line monotonic membership function.

$$w_i = A_{1i}(x_1^0) A_{2i}(x_2^0) A_{3i}(x_3^0). \qquad (13)$$

The defuzzified ith value is

$$\begin{aligned} y_i^* &= \mu_{B_i^{-1}}(w_i) \\ &= \mu_{B_i^{-1}}(A_{i1}(x_1^0) A_{i2}(x_2^0) A_{i3}(x_3^0)). \end{aligned} \qquad (14)$$

We define the overall defuzzified value y^* as Eq. (15) (Note that *weighed combination method* is usually used in Tsukamoto's defuzzification.),

$$y^* = \sum_{i=1}^{8} y_i^*. \qquad (15)$$

This results is equal to that of Eq. (4). We can easily verify that the cases of n-dimensions (one, two, four or more) in MDI produce the same results of previous special Tsukamoto's fuzzy reasoning in which the number of input variable is n.

V. Discussion

We showed two interesting results in this paper. First, multidimensional linear interpolation (MDI) is a special form of Tsukamoto's fuzzy reasoning. Second, if compatibility w_i is well defined, defuzzification strategy can be achieved simply by summing of each defuzzified ith value y_i^*. If we use the followings in Tsukamoto's method, the result is equal to an MDI.

① input variable : fuzzy singleton.
② membership system : STM system as discribed in section IV.
③ algebraic product for compatibility w_i.
④ final overall defuzzified value $y^* = \sum_{i=1}^{N} y_i^*$.

At this point, we need to compare both methods. Even if we can get the same output, an MDI is efficient than fuzzy reasoning because the former uses valid data whereas the later calculates all possible cases of rules even if they produce zero value. If we think input data are contaminated by noise, we can regard input value as fuzzy number when we use fuzzy reasoning method, but an MDI has no flexibility.

VI. Conclusion

We find a multidimensional linear interpolation (MDI) is a special form of Tsukamoto's fuzzy reasoning. From this result, we find the overall defuzzification strategy can be accomplished by adding only each rule's defuzzified value if compatibility w_i is well defined. We compared both MDI and fuzzy reasoning in section V. Further researchs are necessary to find compatibility w_i which can be used in simple defuzzification strategies that the overall defuzzification can be acomplished by adding each defuzzified values as stated before. And there remains the problem of relationship between MDI of tabulated function values at "random" points in n-dimensional space and fuzzy reasoning.

References

[1] A. K. Jain, *Fundamentals of Digital Image Processing*, Prentice-Hall, 1989.

[2] T. Sudkamp and R. J. Hammell II, "Interpolation, Completion, and Learning Fuzzy Rules," *IEEE Trans. on Syst. Man, and Cybern.*, vol. SMC-24, no. 2, pp. 332-342, Feb. 1994.

[3] Y. Tsukamoto, "An Approach to Fuzzy Reasoning Method," in *Advances in Fuzzy Set Theory and Applications*, M. M. Gupta, R. K. Ragada, and R. R. Yager, Eds, Amsterdam : North-Holland, 1979.

[4] W. H. Press, *Numerical Recipes in C*, Cambridge Uverversity Press, 1992.