PAIRWISE FUZZY S-CLOSED BITOPOLOGICAL SPACES

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Abstract: In this paper, we generalize the concept of fuzzy S-closed spaces due to Mukherjee and Ghosh [8] into fuzzy bitopological setting and investigate some of its properties using the concepts of (τ_i, τ_j)-semi-closure and related notions in fuzzy setting.

1. INTRODUCTION AND PRELIMINARIES

Using the concept of fuzzy set, Chang [2] first introduced fuzzy topological spaces. Subsequently many authors [1,8,10-12] continued the investigation of such spaces. Kandil [5] introduced and studied the notion of fuzzy bitopological spaces as a natural generalization of fuzzy topological spaces. Kumar [6,7] defined the concepts of (τ_i, τ_j)-fuzzy semi-open (semi-closed) sets and (τ_i, τ_j)-fuzzy preopen (preclosed) set in the fuzzy bitopological setting.

In this paper, we generalize the concept of fuzzy S-closed spaces due to Mukherjee and Ghosh [8] into fuzzy bitopological setting and discuss some of its properties by using the concepts of (τ_i, τ_j)-semi-closure and related notions in fuzzy setting, and characterize pairwise fuzzy S-closed sets in terms of the concept of fuzzy filterbase. Also the image and inverse image of pairwise fuzzy S-closed spaces under some types of functions are investigated.

For definitions and results not explained in this paper, we refer to the papers [6,7,9,12] assuming them to be well known. A system (X, τ_1, τ_2) consisting of a set X with two fuzzy topologies τ_1 and τ_2 on X is called a fuzzy bitopological space (henceforth, fbts for short) [5]. For a fuzzy set A of a fbts (X, τ_1, τ_2), τ_i-Int(A) and τ_j-Cl(A) means respectively the interior and closure of A with respect to the fuzzy topologies τ_i and τ_j. With τ_i-fo set and τ_j-fc set we shall mean respectively τ_i-fuzzy open set and τ_j-fuzzy closed set. Throughout this paper, the indices i and j take values in $\{1, 2\}$ and $i \neq j$.

Definition 1.1 [6,7,9]. Let A be a fuzzy set in a fbts (X, τ_1, τ_2). Then A is called

(a) (τ_i, τ_j)-fuzzy semi-open (briefly, (τ_i, τ_j)-fso) if there exists a τ_i-fo set B such that $B \leq A \leq \tau_j$-Cl(B),

(b) (τ_i, τ_j)-fuzzy semi-closed (briefly, (τ_i, τ_j)-fsc) if there exists a τ_i-fc set B such that τ_j-Int$(B) \leq A \leq B$,

(c) (τ_i, τ_j)-fuzzy preopen (briefly, (τ_i, τ_j)-fpo) if $A \leq \tau_i$-Int$(\tau_j$-Cl$(A))$,

(d) (τ_i, τ_j)-fuzzy preclosed (briefly, (τ_i, τ_j)-fpc) if τ_i-Cl$(\tau_j$-Int$(A)) \leq A$,

(e) (τ_i, τ_j)-fuzzy regularly open (briefly, (τ_i, τ_j)-fro) if $A = \tau_i$-Int$(\tau_j$-Cl$(A))$,

(f) (τ_i, τ_j)-fuzzy regularly closed (briefly, (τ_i, τ_j)-frc) if $A = \tau_i$-Cl$(\tau_j$-Int$(A))$.

Theorem 1.2 [9]. *For a fuzzy set A in a fbts (X, τ_1, τ_2), the following are equivalent:*
(a) A is (τ_i, τ_j)-fso,
(b) $A \leq \tau_j\text{-}Cl(\tau_i\text{-}Int(A))$,
(c) $\tau_j\text{-}Cl(A) = \tau_j\text{-}Cl(\tau_i\text{-}Int\ (A))$.

Theorem 1.3. *Let A be a fuzzy set in a fbts (X, τ_1, τ_2). Then $x_\alpha \in (\tau_i.\tau_j)\text{-}sCl(A)$ if and only if for each (τ_i, τ_j)-fso set U with $x_\alpha qU$, UqA.*

Theorem 1.4. *If a fuzzy set A in a fbts (X, τ_1, τ_2) is (τ_j, τ_i)-frc, then A is (τ_i, τ_j)-fso.*

Theorem 1.5. *If a fuzzy set A in fbts (X, τ_1, τ_2) is (τ_i, τ_j)-fso, then $\tau_j\text{-}Cl(A)$ is (τ_j, τ_i)-frc.*

2. Pairwise fuzzy S-closed bitopological spaces

Definition 2.1. Let $\mathcal{F} = \{F_\alpha\}$ be a ffb in (X, τ_1, τ_2) and x_α be a fuzzy point in X. Then $\mathcal{F}$ is said to

(a) (τ_i, τ_j)-S-accumulate to x_α if for each (τ_i, τ_j)-fso semi-q-nbd U of x_α and each $F_\alpha \in \mathcal{F}$, $\tau_j\text{-Cl}(U)\text{q}F_\alpha$,

(b) (τ_i, τ_j)-S-converge to x_α if for each (τ_i, τ_j)-fso semi-q-nbd U of x_α, there exists $F_\alpha \in \mathcal{F}$ such that $F_\alpha \leq \tau_j\text{-Cl}(U)$.

Definition 2.2. A fbts (X, τ_1, τ_2) is called (τ_i, τ_j)-fuzzy S-closed if for each fuzzy cover $\{V_\alpha \mid \alpha \in \Lambda\}$ of X with (τ_i, τ_j)-fso sets, there exists a finite subfamily $\{V_{\lambda_k} \mid k = 1, 2, \cdots, n\}$ such that $\cup_{k=1}^{n} \tau_j\text{-Cl}(V_{\lambda_k}) = 1_X$. X is called pairwise fuzzy S-closed if it is (τ_1, τ_2)-fuzzy S-closed and (τ_2, τ_1)-fuzzy S-closed.

Theorem 2.3. *Let $\mathcal{F}$ be an ultra-fuzzy filter in a fbts X. Then $\mathcal{F}$ (τ_i, τ_j)-S-accumulates to a fuzzy point x_α if and only if $\mathcal{F}$ is (τ_i, τ_j)-S-convergent to x_α.*

Theorem 2.4. *The following statements are equivalent for a fbts (X, τ_1, τ_2):*
(a) X is (τ_i, τ_j)-fuzzy S-closed.
(b) Every ultra ffb $\mathcal{F}$ is (τ_i, τ_j)-S-convergent.
(c) Every ffb $\mathcal{B}$ (τ_i, τ_j)-S-accumulates to some fuzzy point in X.
(d) For each family $\{F_\lambda\}$ of (τ_i, τ_j)-fsc sets such that $\cap F_\lambda = 0_X$, there exists a finite subfamily $\{F_{\lambda_1}, F_{\lambda_2}, \cdots, F_{\lambda_n}\}$ of $\{F_\lambda\}$ such that $\cap_{k=1}^{n} \tau_j\text{-}Int(F_{\lambda_k}) = 0_X$.

Definition 2.5. A collection $\mathcal{U}$ of fuzzy sets in a fbts (X, τ_1, τ_2) is said to be a cover of a fuzzy set A if for each $x \in \text{Supp}(A)$, $(\cup\mathcal{U})(x) = 1$. If each member of $\mathcal{U}$ is (τ_i, τ_j)-fso, then $\mathcal{U}$ is called a (τ_i, τ_j)-fuzzy semiopen cover.

Definition 2.6. A fuzzy set A in a fbts (X, τ_1, τ_2) is said to be (τ_i, τ_j)-fuzzy S-closed set (for shortly, (τ_i, τ_j)-FSC set) in X if for every (τ_i, τ_j)-fuzzy semiopen cover of A, there exists a finite subcollection $\mathcal{U}_0$ of $\mathcal{U}$ such that $\cup\{\tau_j\text{-Cl}(U) \mid U \in \mathcal{U}_0\} \geq A$. A fuzzy set A in a fbts (X, τ_1, τ_2) is pairwise fuzzy S-closed set (for shortly, pairwise FSC set) if it is (τ_1, τ_2)-FSC set and (τ_2, τ_1)-FSC set.

Theorem 2.7. *For a fuzzy set A in a fbts (X, τ_1, τ_2), the following statements are equivalent:*

(a) A is (τ_i, τ_j)-FSC set in X.

(b) For each cover $\mathcal{U}$ of A by (τ_j, τ_i)-frc sets, there exists a finite subcollection $\mathcal{U}_0$ of $\mathcal{U}$ such that $(\cup\mathcal{U}_0)(x) = 1$ for all $x \in Supp(A)$.

(c) If $\mathcal{U}$ is a family of (τ_j, τ_i)-fro sets having the property that for every finite subcollection $\mathcal{U}_0$ of $\mathcal{U}$, $(\cap\mathcal{U}_0)(x) > 1 - A(x)$ for some $x \in Supp(A)$, then $\cap\mathcal{U} \neq 0_X$.

(d) For any ffb $\mathcal{B}$ in X by (τ_j, τ_i)-fro sets with $(\cap\mathcal{B}) \cap A = 0_X$, there exists a finite subcollection $\mathcal{B}_0$ of $\mathcal{B}$ such that $(\cap\mathcal{B}_0)\bar{q}A$.

Theorem 2.8. *Let A be a fuzzy set in X. If for each ffb $\mathcal{B}$ in X by (τ_j, τ_i)-fso sets, $[\cap\{(\tau_i, \tau_j)\text{-}sCl(B) \mid B \in \mathcal{B}\}] \cap A = 0_X$ implies there exists a finite subset $\mathcal{B}_0$ of $\mathcal{B}$ such that $\cap\mathcal{B}_0\bar{q}A$, then A is a (τ_i, τ_j)-FSC set in X.*

Theorem 2.9. *A fuzzy set A in a fbts (X, τ_1, τ_2) is (τ_i, τ_j)-FSC set if and only if whenever any fbb $\mathcal{B}$ in X has the property that for any finitely many members $B_1, B_2, \cdots, B_n$ of $\mathcal{B}$ and for any (τ_j, τ_i)-frc set C with $C \geq A$, $(B_1 \cap B_2 \cap \cdots \cap B_n)qC$ holds, then there exists a fuzzy point $x_\alpha \in A$ such that $x_\alpha \in \cap\{(\tau_i, \tau_j)\text{-}sCl(B) \mid B \in \mathcal{B}\}$.*

Theorem 2.10. *If A and B are (τ_i, τ_j)-FSC sets in a fbts (X, τ_1, τ_2), then $A \cup B$ is also (τ_i, τ_j)-FSC set.*

Theorem 2.11. *If A is (τ_i, τ_j)-FSC sets in a fbts (X, τ_1, τ_2), then τ_j-$Cl(A)$ is also (τ_i, τ_j)-FSC set.*

Theorem 2.12. *If A is (τ_j, τ_i)-fro set in a (τ_i, τ_j)-fuzzy S-closed fbts (X, τ_1, τ_2), then A is a (τ_i, τ_j)-FSC set.*

Lemma 2.13 [9]. *A fuzzy set A in a fbts (X, τ_1, τ_2) is (τ_j, τ_i)-fpo if and only if (τ_i, τ_j)-$sCl(A)$ is (τ_j, τ_i)-fro.*

Theorem 2.14. *If A is (τ_j, τ_i)-fpo set in a (τ_i, τ_j)-fuzzy S-closed fbts (X, τ_1, τ_2), then (τ_i, τ_j)-$sCl(A)$ is a (τ_i, τ_j)-FSC set.*

3. Fuzzy pairwise semi-continuous, fuzzy pairwise irresolute functions and fuzzy S-closedness

Definition 3.1. Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a function. Then f is called a

(a) fuzzy pairwise semi-continuous [7] if $f^{-1}(V)$ is (τ_i, τ_j)-fso set of X for each σ_i-fo set V of Y,

(b) fuzzy pairwise irresolute if $f^{-1}(V)$ is (τ_i, τ_j)-fso set of X for each (σ_i, σ_j)-fso set V of Y.

Clearly, every fuzzy pairwise irresolute function is fuzzy pairwise semi-continuous, but the converse need not true. This converse is true if the function is, in addition, fuzzy pairwise open [7].

Lemma 3.2 [7]. *A function* $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ *is fuzzy pairwise semi-continuous if and only if for each fuzzy set A of X,* $f((\tau_i, \tau_j)\text{-}sCl(A)) \leq \sigma_i\text{-}Cl(f(A))$.

Theorem 3.3. *If* $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ *is fuzzy pairwise semi-continuous surjection and* (X, τ_1, τ_2) *is pairwise fuzzy S-closed, then* (Y, σ_1, σ_2) *is pairwise fuzzy almost compact.*

Lemma 3.4. *A function* $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ *is fuzzy pairwise irresolute if and only if for each fuzzy set A of X,* $f((\tau_i, \tau_j)\text{-}sCl(A)) \leq (\sigma_i, \sigma_j)\text{-}sCl(f(A))$.

Theorem 3.5. *Let* (X, τ_1, τ_2) *be PFED [9] and* $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ *be fuzzy pairwise irresolute function. If A is a pairwise FSC set in X, then f(A) is a pairwise FSC set in Y.*

If the set A of above theorem is the whole set X, then we can drop the condition that (X, τ_1, τ_2) is PFED.

Theorem 3.6. *If* $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ *is fuzzy pairwise irresolute surjection and* (X, τ_1, τ_2) *is pairwise fuzzy S-closed, then* (Y, σ_1, σ_2) *is also pairwise fuzzy S-closed.*

Theorem 3.7. *Let* $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ *be fuzzy pairwise semi-continuous and* $f : (X, \tau_2) \to (Y, \sigma_2)$ *is fuzzy continuous and fuzzy open. If A is* (τ_1, τ_2)*-FSC set in X, then f(A) is* (σ_1, σ_2)*-FSC set in Y.*

Corollary 3.8. *Let* $(X_1, \tau_1^1, \tau_2^1)$ *and* $(X_2, \tau_1^2, \tau_2^2)$ *be fbts's such that* X_1 *is product related to* X_2 *[1]. If the product fbts* $(X_1 \times X_2, \sigma_1, \sigma_2)$ *is pairwise fuzzy S-closed, where* σ_k *is the fuzzy product topology [11] generated by* τ_k^1 *and* τ_k^2 $(k = 1, 2)$*, then each* $(X_i, \tau_1^i, \tau_2^i)$ $(i = 1, 2)$*is also pairwise fuzzy S-closed.*

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