PRODUCTS OF T–GENERALIZED STATE MACHINES AND T–GENERALIZED TRANSFORMATION SEMIGROUPS

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Abstract: We investigate some of algebraic properties of T-generalized state machines, T-generalized transformation semigroups, coverings of T-generalized state machines and T-generalized transformation semigroups.

1. Introduction

Since Wee [6] in 1967 introduced the concept of fuzzy automata following Zadeh [7], fuzzy automata theory has been developed by many researchers. Recently Malik et al. [2-4] introduced the concepts of fuzzy state machines and fuzzy transformation semigroups based on Wee's concept of fuzzy automata and related concepts and applied algebraic technique. In this paper, we introduce the concepts of T-generalized state machines and T-generalized transformation semigroups that are different from their concepts, coverings of T-generalized state machines and T-generalized transformation semigroups that are generalizations of crisp concepts in algebraic aotomata theory and investigate their algebraic structures.

For the terminology in (crisp) algebraic automata theory, we refer to [1].

2. T-generalized state machines

DEFINITION 2.1. *A triple $\mathcal{M} = (Q, X, \tau)$ where Q and X are finite nonempty sets and τ is a fuzzy subset of $Q \times X \times Q$, i.e., τ is a function from $Q \times X \times Q$ to $[0,1]$, is called a generalized state machine if $\sum_{q \in Q} \tau(p, a, q) \leq 1$ for all $p \in Q$ and $a \in X$. If $\sum_{q \in Q} \tau(p, a, q) = 1$ for all $p \in Q$ and $a \in X$, then $\mathcal{M}$ is said to be complete.*

DEFINITON 2.2 [5]. *A binary operation T on $[0,1]$ is called a t-norm if*

(1) $T(a, 1) = a$,

(2) $T(a, b) \leq T(a, c)$ whenever $b \leq c$,

(3) $T(a, b) = T(b, a)$,

(4) $T(a, T(b, c)) = T(T(a, b), c)$

for all $a, b, c \in [0,1]$.

By an abuse of notation we will denote $T(a_1, T(a_2, T(\cdots, T(a_{n-1}, a_n)\cdots)))$ by $T(a_1, \cdots, a_n)$ where $a_1, \cdots, a_n \in [0,1]$. The legitimacy of this abuse is ensured by the associativity of T (Definition 2.2(4)).

Let $\mathcal{M} = (Q, X, \tau)$ be a generalized state machine. Then Q is called the set of states and X is called the set of input symbols. Let X^+ denote the set of all words of elements of X of finite length.

DEFINITION 2.3. *Let $\mathcal{M} = (Q, X, \tau)$ be a generalized state machine. Define $\tau^+ : Q \times X^+ \times Q \longrightarrow [0,1]$ by*

$$\begin{aligned}&\tau^+(p, a_1 \cdots a_n, q) \\ &= \vee \{T(\tau(p, a_1, r_1), \tau(r_1, a_2, r_2), \cdots, \tau(r_{n-2}, a_{n-1}, r_{n-1}), \tau(r_{n-1}, a_n, q)) | r_i \in Q\}\end{aligned}$$

where $p, q \in Q$ and $a_1, \cdots, a_n \in X$. When T is applied to $\mathcal{M}$ as above, $\mathcal{M}$ is called a T-generalized state machine.

Hereafter a generalized state machine will always be written as a T-generalized state machine because a generalized state machine always induces a T-generalized state machine as in Definition 2.3.

For a T-generalized state machine, $\equiv$ is a relation on X^+ defined bt $x \equiv y$ if $\tau^+(p, x, q) = \tau^+(p, y, q)$ for all $p, q \in Q$.

Given a T-generalized state machine $\mathcal{M} = (Q, X, \tau)$, we will write $\{y \in X^+ | x \equiv y\}$ by $[x]$ where $x \in X^+$ and $X^+/\equiv = \{[x] | x \in X^+\}$ by $S(\mathcal{M})$.

THEOREM 2.4. *Let $\mathcal{M} = (Q, X, \tau)$ be a T-generalized state machine. Then $S(\mathcal{M})$ is a semigroup, where the binary operation on $S(\mathcal{M})$ is defined by $[x][y] = [xy]$.*

3. T-generalized transformation semigroups

We now generalize the concept of a transformation semigroup.

DEFINITION 3.1. *A T-generalized state machine (Q, S, ρ) is called a T-generalized transformation semigroup if S is a finite semigroup and if it satisfies the following:*

(1) $\rho(p, uv, q) = \vee\{T(\rho(p, u, r), \rho(r, v, q)) | r \in Q\}$ for all $p, q \in Q$ and $u, v \in S$.

(2) For $u, v \in S$, if $\rho(p, u, q) = \rho(p, v, q)$ for all $p, q \in Q$, then $u = v$.

When a T-generalized transformation semigroup $\mathcal{S} = (Q, S, \rho)$ is regarded as a T-generalized state machine (Q, S, τ_ρ) by taking $\tau_\rho = \tau_\rho^+ = \rho$, we will write it by $SM(\mathcal{S})$.

DEFINITION 3.2. *A T-generalized transformation semigroup (Q, S, ρ) is called a T-generalized transformation monoid if S is a monoid with identity e and it satisfies the following:*

$$\rho(p, e, q) = \begin{cases} 1 & \text{if } p = q \\ 0 & \text{if } p \neq q. \end{cases}$$

A T-generalized transformation monoid (Q, S, ρ) is called a T-generalized transformation group if S is a group.

DEFINITION 3.3. *A t-norm T is said to be T-generalized transformation seimgroup inducible if $S(\mathcal{M})$ is finite and $\sum_{q \in Q} \tau^+(p, x, q) \leq 1$ for all $p \in Q$ and $x \in X^+$ for every T-generalized state machine $\mathcal{M} = (Q, X, \tau)$.*

Actually, let T be T-generalized transformation semigroup inducible, then a T-generalized state machine $\mathcal{M} = (Q, X, \tau)$ naturally induces a T-generalized transformation semigroup $(Q, S(\mathcal{M}), \rho_\tau)$, where ρ_τ is defined by $\rho_\tau(p, [x], q) = \tau^+(p, x, q)$ by Theorem 2.4. We call

$(Q, S(\mathcal{M}), \rho_\tau)$ by the T-generalized transformation semigroup induced by $\mathcal{M}$ and denote it by $TS(\mathcal{M})$.

PROPOSITION 3.4. *There exists a T-generalized transformation semigroup inducible t-norm T.*

From now on, we always assume that T is T-generalized transformation semigroup inducible whenever we deal with T-generalized transformation semigroups induced by T-generalized state machines.

4. Coverings

DEFINITION 4.1. *Let $\mathcal{M}_1 = (Q_1, X_1, \tau_1)$ and $\mathcal{M}_2 = (Q_2, X_2, \tau_2)$ be T-generalized state machines. If $\xi : X_1 \longrightarrow X_2$ is a function and $\eta : Q_2 \longrightarrow Q_1$ is a surjective partial function such that $\tau_1^+(\eta(p), x, \eta(q)) \leq \tau_2^+(p, \xi(x), q)$ for all p, q in the domain of η and $x \in X_1^+$, then we say that (η, ξ) is a covering of $\mathcal{M}_1$ by $\mathcal{M}_2$ and that $\mathcal{M}_2$ covers $\mathcal{M}_1$ and denote by $\mathcal{M}_1 \leq \mathcal{M}_2$. Moreover, if the inequality turns out equality whenever the left hand side of the inequality is not zero [resp. the inequality always turns out equality], then we say that (η, ξ) is a strong covering [resp. a complete covering] of $\mathcal{M}_1$ by $\mathcal{M}_2$ and that $\mathcal{M}_2$ strongly covers [resp. completely covers] $\mathcal{M}_1$ and denote by $\mathcal{M}_1 \leq_s \mathcal{M}_2$ [resp. $\mathcal{M}_1 \leq_c \mathcal{M}_2$].*

In Definition 4.1, we abused the function ξ. We will write the natural semigroup homomorphism from X_1^+ to X_2^+ induced by ξ by ξ also for convenience sake. We give an example that is elementary and important.

EXAMPLE 4.2. *Let $\mathcal{M} = (Q, X, \tau)$ be a T-generalized state machine. Define an equivalence relation $\sim$ on X by $a \sim b$ if and only if $\tau(p, a, q) = \tau(p, b, q)$ for all $p, q \in Q$. Construct a T-generalized state machine $\mathcal{M}_1 = (Q, X/\sim, \tau^\sim)$ by defining $\tau^\sim(p, [a], q) = \tau(p, a, q)$. Now define $\xi : X \longrightarrow X/\sim$ by $\xi(a) = [a]$ and $\eta = 1_Q$. Then (η, ξ) is a complete covering of $\mathcal{M}$ by $\mathcal{M}_1$ clearly.*

DEFINITION 4.3. *Let $\mathcal{S}_1 = (Q_1, S_1, \rho_1)$ and $\mathcal{S}_2 = (Q_2, S_2, \rho_2)$ be T-generalized transformation semigroups. If $\eta : Q_2 \longrightarrow Q_1$ is a surjective partial function and for each $s \in S_1$ there exists $t_s \in S_2$ such that $\rho_1(\eta(p), s, \eta(q)) \leq \rho_2(p, t_s, q)$ for all p, q in the domain of η, then we say that η is a covering of $\mathcal{S}_1$ by $\mathcal{S}_2$ and that $\mathcal{S}_2$ covers $\mathcal{S}_1$ and denote by $\mathcal{S}_1 \leq \mathcal{S}_2$. Moerover, if the inequality turns out equality whenver the left hand side of the inequality is not zero [resp. the inequality always turns out equality], then we say that η is a strong covering [resp. a complete covering] of $\mathcal{S}_1$ by $\mathcal{S}_2$ and that $\mathcal{S}_2$ strongly covers [resp. completely covers] $\mathcal{S}_1$ and denote by $\mathcal{S}_1 \leq_s \mathcal{S}_2$ [resp. $\mathcal{S}_1 \leq_c \mathcal{S}_2$].*

THEOREM 4.4. *Let $\mathcal{M}_1 = (Q_1, X_1, \tau_1)$ and $\mathcal{M}_2 = (Q_2, X_2, \tau_2)$ be T-generalized state machines such that $\mathcal{M}_1 \leq \mathcal{M}_2$ with covering (η, ξ). Then $TS(\mathcal{M}_1) \leq TS(\mathcal{M}_2)$. Moreover, if $\mathcal{M}_1 \leq_c \mathcal{M}_2$ and η is a function, then $TS(\mathcal{M}_1) \leq_c TS(\mathcal{M}_2)$.*

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