FUZZY CONVERGENCE

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In this paper, we introduce two fuzzy convergence structures, fuzzy convergence and fuzzy limitierung, and obtain a relationship between them. We also consider relationships between fuzzy limit space and pseudotopological convergence space.

1 Fuzzy convergence space

Let X be a set and $\mathfrak{P}(X)$ be the collection of all prefilters on X.

Definition 1.1 *A fuzzy convergence space* is a pair $(X, \lim)$, where $X \in$ | SET | and where

$$\lim : \mathfrak{P}(X) \to I^X : \mathfrak{F} \to \lim \mathfrak{F}$$

satisfies the conditions:

(PST) $\forall \mathfrak{F} \in \mathfrak{P}(X) : \lim \mathfrak{F} = \inf_{\mathfrak{G} \in \mathfrak{P}_m(\mathfrak{F})} \lim \mathfrak{G}$.

(F1p) $\forall \mathfrak{F} \in \mathfrak{P}_p(X) : \lim \mathfrak{F} \leq c(\mathfrak{F})$

(F2p) $\forall \mathfrak{F}_1, \mathfrak{F}_2 \in \mathfrak{P}_p(X) : \mathfrak{F}_1 \subset \mathfrak{F}_2 \Rightarrow \lim \mathfrak{F}_2 \leq \lim \mathfrak{F}_1$.

(C1) $\forall x \in X, \forall \alpha \in I_0$: $\lim \alpha \dot{1}_x \geq \alpha 1_x$.

where $I_0 = (0, 1]$. Whenever necessary, to avoid confusion, we may write $\lim_X$ or similar instead of lim.

Definition 1.2 A function $f : (X, \lim_X) \to (Y, \lim_Y)$ is said to be continuous if for every $\mathfrak{F} \in \mathfrak{P}(X)$ and $x \in X$, $\lim_X \mathfrak{F}(x) \leq \lim_Y f(\mathfrak{F})(f(x))$.

We denote by **FCS** the concrete category with objects all fuzzy convergence spaces and morphisms all continuous maps. Then, the category **FTS** of fuzzy topological spaces, is nicely embedded in **FCS**.[7]

Theorem 1.3 ***FTS*** *is a bireflective subcategory of* ***FCS***.

2 Fuzzy limitierung

Let X be a set, $\mathfrak{P}(X) =$ the collection of all prefilters on X and $\mathcal{H} =$ the set of all fuzzy points in X.[9]

Definition 2.1 A *fuzzy limitierung* Δ is a map from $\mathcal{H}$ into $\mathcal{P}(\mathfrak{P}(X))$, the power set of $\mathfrak{P}(X)$, subject to the following axioms: for each $\mathfrak{p} = (x, \lambda)$,

(L0) $\mathfrak{F} \in \Delta(\mathfrak{p}) \Rightarrow \underline{\alpha} \in \mathfrak{F}$ for all $\alpha > 1 - \lambda$

(L1) $\langle \mathfrak{p} \rangle = \{\mu \in I^X \mid \mathfrak{p} \text{ q } \mu\} \in \Delta(\mathfrak{p})$

(L2) $\mathfrak{F} \in \Delta(\mathfrak{p})$ and $\mathfrak{F} \subseteq \mathfrak{G} \Rightarrow \mathfrak{G} \in \Delta(\mathfrak{p})$

(L3) $\mathfrak{F}, \mathfrak{G} \in \Delta(\mathfrak{p}) \Rightarrow \mathfrak{F} \cap \mathfrak{G} \in \Delta(\mathfrak{p})$

The pair (X, Δ) is called a *fuzzy limit space*. If $\mathfrak{F} \in \Delta(\mathfrak{p})$, we say that $\mathfrak{F}$ *converges* to $\mathfrak{p}$ and $\mathfrak{p}$ is a limit of $\mathfrak{F}$ with respect to Δ.

Definition 2.2 A function $f : (X, \Delta_X) \to (Y, \Delta_Y)$ is said to be fuzzy continuous if for every $\mathfrak{p} \in \mathcal{H}$, and for every $\mathfrak{F} \in \Delta_X(\mathfrak{p})$, $f(\mathfrak{F}) \in \Delta_Y(f(\mathfrak{p}))$.

We denote by **FLim** the concrete category with objects all fuzzy limit spaces and morphisms all fuzzy continuous maps. Then, **FTS** is embedded in **FLim**. [9]

Theorem 2.3 ***FTS** is a bireflective subcategory of **FLim**.*

3 Relation between FCS and FLim

In this chapter, we prove main theorems, that is, **FCS** is embedded in **FLim**. Furthermore, we investigate some relationships between **FLim** and the category of pseudotopological convergence spaces.

Theorem 3.1 *If (X, Δ) is a fuzzy limit space, then the map*

$$\mathfrak{P}(X) \to I^X : \mathfrak{F} \to \lim_\Delta \mathfrak{F}$$

where

$$\lim_\Delta \mathfrak{F}(x) = \inf\{1 - \lambda \mid \mathfrak{F} \in \Delta((x, \lambda))\} \quad \forall \mathfrak{F} \in \mathfrak{P}_p(X)$$
$$\lim_\Delta \mathfrak{F}(x) = \inf_{\mathfrak{G} \in \mathfrak{P}_m(\mathfrak{F})} \lim_\Delta \mathfrak{G}(x) \qquad \forall \mathfrak{F} \in \mathfrak{P}(X)$$

satisfies the conditions (PST), (F1p), (F2p) and (C1) in the Definition 1.1.

Proposition 3.2 Let (X, Δ_X), (Y, Δ_Y) be fuzzy limit spaces. Suppose that $f : (X, \Delta_X) \to (Y, \Delta_Y)$ is a fuzzy continuous map in **FLim**. Then $f : (X, \lim_{\Delta_X}) \to (Y, \lim_{\Delta_Y})$ is a continuous map in **FCS**.

Theorem 3.3 *Let $(X, \lim)$ be a fuzzy convergence space. Define*

$$\Delta_{\delta_{\lim}} : \mathcal{H} \longrightarrow \mathcal{P}(\mathfrak{P}(X))$$

such that for any $\mathfrak{p} \in \mathcal{H}$,

$$\mathfrak{F} \in \Delta_{\delta_{\lim}}(\mathfrak{p}) \text{ iff } \mathcal{N}((x, \lambda)) \subset \mathfrak{F} \quad \forall \mathfrak{F} \in \mathfrak{P}(X)$$

where $\{\nu \in I^X \mid \exists \omega \in \delta_{\lim}$ such that $1 - \lambda < \omega(x),\ \omega(x) \le \nu(x)\}$. Then, $(X, \Delta_{\delta_{\lim}})$ is a fuzzy limit space.

Proposition 3.4 Let $(X, \lim_X)$, $(Y, \lim_Y)$ be fuzzy convergence spaces, and let $f : (X, \lim_X) \to (Y, \lim_Y)$ be a continuous map in **FCS**. Then $f : (X, \Delta_{\delta_{\lim_X}}) \to (Y, \Delta_{\delta_{\lim_Y}})$ is a fuzzy continuous map in **FLim**.

By above results we can define functors L : **FLim** $\to$ **FCS** such that $L(X,\Delta)=(X,\lim_\Delta)$, $L(f)=f$ and R : **FCS** $\to$ **FLim** such that $R(X,\lim)=(X,\Delta_{\lim})$, $R(f)=f$, where we denote $\Delta_{\delta\lim}$ by $\Delta_{\lim}$. Then, we have following result.

Proposition 3.5 (1) For any fuzzy point $\mathfrak{p}$ in X, $\Delta(\mathfrak{p})\subset\Delta_{\lim_\Delta}(\mathfrak{p})$

(2) For any prefilter $\mathfrak{F}$ in X, $\lim(\mathfrak{F})=\lim_{\Delta_{\lim}}(\mathfrak{F})$

Theorem 3.6 *FCS is a bireflective subcategory of* ***FLim***.

4 Relation between FLim and PSTOP

Another familiar category which can be embedded in **FLim** is **PSTOP**. We recall the definition [8]. By a *pseudotopological convergence structure* on a set X, we mean a map

$$q : X \longrightarrow \mathcal{P}(\mathcal{F}(X))$$

which satisfies the following axioms:

(PC1) $\dot{x}\in q(x)$

(PC2) $\mathcal{F}\in q(x),\mathcal{G}\subset\mathcal{F}\Rightarrow\mathcal{G}\in q(x)$

(PC3) $(\forall\mathcal{U}(\mathcal{F}):\mathcal{F}\in q(x))\Rightarrow\mathcal{F}\in q(x)$

where $\mathcal{F}(X)=$ the set of all filters on X and $\mathcal{U}(\mathfrak{F})=$ the set of all ultrafilters finer than $\mathfrak{F}$.

To prove that **PSTOP** is embedded in **FLim**, we use the facts **PSTOP** is a bireflective subcategory of **Lim** [8] and **Lim** is a bicoreflective subcategory of **FLim** [9]. Recall the definition of **Lim** [8]. A *limitierung* q on a set X is a map from X to $\mathcal{P}(\mathcal{F}(X))$, where $\mathcal{F}(X)$ is the collection of all filters on X, subjects to the following axioms : for each $x\in X$,

(C1) $\langle x\rangle=\{A\subset X\mid x\in A\}\in q(x)$

(C2) $\mathcal{F}\in q(x)$ and $\mathcal{F}\subseteq\mathcal{G}\Rightarrow\mathcal{G}\in q(x)$

(C3) $\mathcal{F},\ \mathcal{G}\in q(x)\Rightarrow\mathcal{F}\cap\mathcal{G}\in q(x)$

The pair (X,q) is called a *limit space*. We sometimes write $\mathcal{F}\to x$ instead of $\mathcal{F}\in q(x)$. Let (X,q_X), (Y,q_Y) be limit spaces. Then $f:(X,q_X)\to(Y,q_Y)$ is *continuous* in **Lim** if and only if $\mathcal{F}\in q_X(x)\Rightarrow f(\mathcal{F})\in q_Y(f(x))\quad\forall x\in X$. E. Lowen showed [8] that if (X,q) is a limit space, then the **PSTOP** reflection of (X,q) is $\mathrm{id}_X:(X,q)\to(X,\tilde{q})$ where $(X,\tilde{q})$ is the limit space defined by :

$$\mathfrak{F}\in\tilde{q}(x)\quad\text{iff}\quad\forall\mathcal{U}\in\mathcal{U}(\mathcal{F}),\quad\mathcal{U}\in q(x)$$

and Min showed [9] that if (X,Δ) is a fuzzy limit space, then $\mathrm{id}_X:(X,\Delta_{q_\Delta})\to(X,\Delta)$ is a bicoreflection of (X,Δ), where q_Δ is a limit structure defined by $\mathcal{F}\in q_\Delta(x)$ iff $\forall\lambda\in(0,1]$, there exists $\mathfrak{F}^\lambda{}_\mathcal{F}\in\Delta((x,\lambda))$ with a basis $\mathfrak{B}$ such that for every $\nu\in\mathfrak{B},\nu(\mathcal{F})\to\nu(x)$ in I_r and $\nu(x)>1-\lambda$, and Δ_q is a fuzzy limit structure defined by

$$\Delta_q(\mathfrak{p})=\{\mathfrak{F}\in\mathfrak{P}(X)\mid\mathfrak{F}\supset\mathfrak{G}^\lambda{}_\mathcal{F}\text{ for some }\mathcal{F}\in q(x)\}$$

where $\mathfrak{G}^\lambda{}_\mathcal{F}=\langle\{\mu\in I^X\mid\mu(\mathfrak{F})\to\mu(x)\text{ in }I_r,\ \mu(x)>1-\lambda\}\rangle$.

Theorem 4.1 *Let (X, q) be a pseudotopological convergence space, then the functor P : **PSTOP** $\rightarrow$ **FLim** defined by $P(X, q) = (X, \Delta_q)$ and $P(f) = f$ is an embedding of **PSTOP** in **FLim**.*

Theorem 4.2 ***PSTOP** is a simultaneously bireflective and bicoreflective subcategory of **FLim**.*

Lowen showed that **PSTOP** is a simultaneously bireflective and bicoreflective subcategory of **FCS** [7]. By the above Theorem, we have a similar result as to **FLim**.

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