Group G-compatible fuzzy matrices

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Let G= $\{g_1 = e, g_2, \cdots, g_n\}$ be a nontrivial finite multiplicative group of order n and let I_n={1, 2, ⋯, n}, n∈ N. We put $g_i g_j = g_{i*j}$,then $< I_n, *>$ is a group isomorphic to G. Also, R=(r_{ij}) will always denote an $n \times n$ fuzzy matrix with elements in [0, 1].

We put $r_{1i} = a_i$ and $i^{-1} = \underline{i}$, for all $i \in I_n$

Definition [1, 2, 3, 4]

The following operations are defined for all $x, y, \alpha \in [0,1]$

(1) $x + y = x \vee y$

(2) $xy = x \wedge y$

(3) $x\,\theta\, y = \begin{cases} x & \text{if } x > y \\ 0 & \text{if } x \le y \end{cases}$

(4) $x \leftarrow y = \begin{cases} 1 & \text{if } x \ge y \\ y & \text{if } x < y \end{cases}$

(5) $x^{\alpha} = \begin{cases} 1 & \text{if } x \ge \alpha \\ 0 & \text{if } x < \alpha \end{cases}$

(6) $x_{\alpha} = \begin{cases} x & \text{if } x \ge \alpha \\ 0 & \text{if } x < \alpha \end{cases}$

(7) $x_c = 1 - x$

Definition [1, 2, 3, 4]

The following operations are defined on the fuzzy matrices

$A = (a_{ij})_{m \times n}$, $B = (b_{ij})_{m \times p}$, $D = (d_{ij})_{p \times q}$, $G = (g_{ij})_{m \times n}$ and $R = (r_{ij})_{n \times n}$

(1) $A + G = (a_{ij} + g_{ij})$

(2) $A \wedge G = (a_{ij}\, g_{ij})$

(3) $BD = (C_{ij})$, where $C_{ij} = \sum_{k=1}^{p} b_{ik} d_{kj}$

(4) $R^c = (r_{ij}^c)$

(5) $A\,\theta\,G = (a_{ij}\,\theta\,g_{ij})$

(6) $B \leftarrow D = (s_{ij})$, where $s_{ij} = \prod_{k=1}^{p}(b_{ik} \leftarrow d_{kj})$

(7) $A \le G$ *iff* $a_{ij} \le g_{ij}$ for all i, j

(8) $A_\alpha = (a_{ij\alpha})$

(9) $A^\alpha = (a_{ij}^\alpha)$

Note that A_α is called the α-level of and $A_0 = A$

Definition [6]

$R = (r_{ij})_{n\times n}$

$|R| = \sum_{\pi\in S_n} r_{1\pi(1)} r_{2\pi(2)} \cdots R_{n\pi(n)}$ where S_n is the symmetric group over the set I_n

Definition [6]

The adjoint of R is defined to be the fuzzy matrix $ad_j\,(R) = (s_{ij})_{n\times n}$, such that $s_{ij} = |R_{ji}|$, where $|R_{ji}|$ is the determinant of (n-1)×(n-1) fuzzy matrix obtained from R by deleting row j and column i. One can obtain the elements of adj (R)=(s_{ij}) as follows : $s_{ij} = \sum_{\pi\in S_{njni}} \prod_{t\in n_j} r_{t\pi(t)}$, where $n_i = I_n \setminus \{i\}$ and s_{njni} is the set of all permutations of the set n_j over the set n_i.

Definition [1]

Let A be a fuzzy subset of a group G (need not to be finite) then A is called fuzzy subgroup of G iff $A(xy) \ge A(x) \wedge A(y), A(x) = A(x^{-1})$ and A(e)=1 for all $x, y \in$G

Definition [14]

R is called left(right) G-compatible fuzzy matrix iff $R_{ij} \le R_{k*i,\,k*j}$ $(r_{ij} \le r_{i*k,\,j*k})$ for all $i, j, k \in$ I_n.

It must be noted that the following results (for left G-compatible fuzzy matrix) are also true for right G-Compatible fuzzy matrix.

Remark

R is left G-compatible iff $r_{ij} = r_{k*i,k*j,}$ for all $i, j, k \in I_n$. This implies that the slsments of the diagonal are equal.

Proposition 1

The following statements are equivalent

(1)R is left G-compatible

(2) $r_{ij} = a_{i*j}, \forall i \in I_n$

(3) R^a is left G-compatible

(4) R_a is left G-compatible

Proposition 2

Let G=<g>={ $g_i \mid g_i = g^{i-1}, i \in I_n$}. Then R is left G-compatible if and only if $r_{ij} = r_{k\oplus i, k\oplus j,} \forall i,j,k \in I_n$, where $\oplus$ is the sum modulo n.

<u>Proof</u>

($\Rightarrow$) Let R he left G-compatible and $i, j, k \in I_n$.

Then $g_{i\oplus k} = g_{i+k} = g^{i+k-i} = g_i g_k g_2 = g_{i*j*2}$. So we have $i\oplus k = i*j*2$.

Since $*$ is communicative, we have $r_{ij} = r_{(k*2)*i,(k*2)*j} = r_{i\oplus k, j\oplus k}$.

($\Rightarrow$) Let $r_{ij} = r_{i\oplus k, j\oplus k}$, $i, j, k \in I_n$. Since $n*2 = 1$.

We have $r_{ij} = r_{i\oplus(k*n), j\oplus(k*n)} = r_{i*(k*n)*2, j*(k*n)*2} = r_{k*i,k*j}$ then R is left G-compatible

Example 1

Let G he the four group. Then the table of (I_4,$*$) is as follows :

	1	2	3	4
1	1	2	3	4
2	2	1	4	3
3	3	4	1	2
4	4	3	2	1

Since $\underline{i}= i$, $\forall i \in I_4$, it follows from proposition that

$$R=\begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \alpha_2 & \alpha_1 & \alpha_4 & \alpha_3 \\ \alpha_3 & \alpha_4 & \alpha_1 & \alpha_2 \\ \alpha_4 & \alpha_3 & \alpha_2 & \alpha_1 \end{pmatrix}, \alpha_1, \alpha_2, \alpha_3, \alpha_4 \in [0,1]$$

is a left G-compatible fuzzy matrix. Note that R is also right G-compatible, since G is commutative.

For fuzzy symmetric groups ([ ]) this may he of some help in reading this paper. We consider the group G as a symmetric group S_n. In this case it is deeply connected with Cayley-table matrices. Then we define G-compatible fuzzy matrices.

Example 2

Let S_3 he the symmetric group over I_3.

We let $e=g_1, (1,2)=g_2, (1,3)=g_3, (2,3)=g_4, (1,2,3)=g_5$ and $(1,3,2)=g_6$ the following is a (I_6) multiplication table.

*	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	1	5	6	3	4
3	3	6	1	5	4	2
4	4	5	6	1	2	3
5	5	3	4	2	1	5
6	6	4	2	3	6	1

Let $H = \{h(i) \in [0,1] \mid i \in I_6\}$. We define a fuzzy matrix $A=(a_{ij})$ by the following.

$$A=(a_{ij})=\begin{pmatrix} h(1) & h(2) & h(3) & h(4) & h(5) & h(6) \\ h(2) & h(1) & h(5) & h(6) & h(3) & h(4) \\ h(3) & h(6) & h(1) & h(5) & h(4) & h(2) \\ h(4) & h(5) & h(6) & h(1) & h(2) & h(3) \\ h(6) & h(3) & h(4) & h(2) & h(1) & h(5) \\ h(5) & h(4) & h(2) & h(3) & h(6) & h(1) \end{pmatrix}$$

We can check that A is a S_3-compatible fuzzy matrix.

Definition

Let G he a group. A multiplication table of G will be a Cayley-table of G.

Example 3

Let B= (b_{ij}) he a fuzzy matrix defined by the following.

$$B=(a_{ij})=\begin{pmatrix} h(1) & h(2) & h(3) & h(4) & h(5) & h(6) \\ h(2) & h(1) & h(5) & h(6) & h(3) & h(4) \\ h(3) & h(6) & h(1) & h(5) & h(4) & h(2) \\ h(4) & h(5) & h(6) & h(1) & h(2) & h(3) \\ h(6) & h(3) & h(4) & h(2) & h(1) & h(5) \\ h(5) & h(4) & h(2) & h(3) & h(6) & h(1) \end{pmatrix}$$

This matrix B is not S_3-compatible.

Definition [6]

(i) A^t denotes the transpose of a matrix.

(ii) Let $A=(a_{ij})$ he a matrix. For a_{ij}, we say that (i,j) is the coordinate of a_{ij}. This definition will he helpful to understand the concept of G-compatible fuzzy matrices.

Proposition 3

Let G he a finite group of order n and let $A=(a_{ij})$ he an n by n fuzzy matrix. Then the following statements are eqivalent.

(i) A is a G-compatible fuzzy matrix.

(ii) $A^t=B=(b_{ij})$ is a G-compatible fuzzy matrix

(iii) $a_{ij}=h(\underline{i}*j)$ for all $i,j,k\in \mathrm{I}_n$, where $h(k)$ is defined by $a_{1i}=h(k)$, and $*$ is the operation of the coordinate group n(G).

(iv) Let n(G) he the coordinate group of G. We define $a_{1i}=h(i)$.

If $a_{ij}=h(k)$ then $a_{ji}=h(\underline{k})$, for all $i,j,k\in \mathrm{I}_n$, and $\underline{k}$ denotes the invers of k in $n(G)$.

Proof

(i) implies (iii). We assume that A is G-compatible and that $a_{k*ik*j} = a_{ij}$ for all $i, j, k \in I_n$. We put $k = \underline{i}$ in $a_{k*ik*j} = a_{ij}$ and we obtain that $a_{k*ik*j} = a_{1i*j} = h(\underline{i}*j)$ and $a_{ij} = h(\underline{i}*j)$. This proves the implication of (i)→(iii), (iii)→(i), (i)→(ii), (iii)→(iv) and (iv)→(iii).

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