On Fuzzy Strongly Continuous Mappings.

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1.Introduction.

We introduce the concept of fuzzy strong continuity and investigate some properties related to the fuzzy strong continuity.

2. Preliminiaries.

In this sectin, we will introduce some elementary concepts and results in fuzzy set theory. Also we will introduce some elementary concepts and results in fuzzy topological spaces.
Throughout this paper, we will denote the unit interval [0,1] of the real line as I. X, Y and Z, etc, will denote sets. For X, I^X denotes the collection of all the mapping from X into I.

Definition 2.1[2]. *Let X be a non - empty set. Then A is called a* **fuzzy set** *in X if $A \in I^X$.*

$\underline{0}_X$ (simply, $\underline{0}$) and $\underline{1}_X$ (simply, $\underline{1}$) denote the empty fuzzy set and the hole fuzzy set in X given by $\underline{0}_X(x) = 0$ for all $x \in X$ and $\underline{1}_X(x) = 1$ for all $x \in X$, respectively.

Definition 2.2[2]. *Let $f : X \to Y$ be a mapping and let $A \in I^X$ and $B \in I^Y$.*

(1) *The* **inverse image** *of B under f, denoted by $f^{-1}(B)$, is a fuzzy set in X defined by for all $x \in X$,*

$$[f^{-1}(B)](x) = B(f(x)) = (B \circ f)(x).$$

(2) *The* **image** *of A under f, denoted by $f(A)$, is a fuzzy set in Y defined by for each $y \in Y$,*

$$[f(A)](y) = \begin{cases} Sup_{x \in f^{-1}(y)} A(x) & \text{if } f^{-1}(y) \neq \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

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where $f^{-1}(y) = \{x \in X : f(x) = y\}$.

Lemma 2.A[2]. *Let $f : X \to Y$ be a mapping. Then:*

(1) $f^{-1}(B^c) = [f^{-1}(B)]^c$ *for each* $B \in I^Y$.
(2) $f(A^c) \supset [f(A)]^c$ *for each* $A \in I^X$.
(3) *If* $B_1 \subset B_2$, *then* $f^{-1}(B_1) \subset f^{-1}(B_2)$, *where* $B_1, B_2 \in I^Y$.
(4) *If* $A_1 \subset A_2$, *then* $f(A_1) \subset f(A_2)$, *where* $A_1, A_2 \in I^X$.
(5) $B \supset f(f^{-1}(B))$ *for each* $B \in I^Y$.
(6) $A \subset f^{-1}(f(A))$ *for each* $A \in I^X$.
(7) *Let* $g : Y \to Z$ *be any mapping. Then* $(g \circ f)^{-1}(C) = f^{-1}(g^{-1}(C))$ *for each* $C \in I^Z$.

We can easily obtain the following results :

Lemma 2.B[1]. *Let $f : X \to Y$ be mapping.*

(1) *If f is injective, then* $f^{-1}(f(A)) = A$ *for each* $A \in I^X$.
(2) *If f is surjective, then* $f(f^{-1}(B)) = B$ *for each* $B \in I^Y$.
(3) *If f is bijective, then* $[f(A)]^c = f(A^c)$.

Definition 2.3 [2,3,5]. *Let X be a non-empty set. Then a subset $\mathcal{T}$ of I^X is called a* **fuzzy topology** *on X if $\mathcal{T}$ satisfies the following conditions:*

(1) $\underline{0}_X, \underline{1}_X \in \mathcal{T}$.
(2) *If* $A, B \in \mathcal{T}$, *then* $A \cap B \in \mathcal{T}$.
(3) *If* $\{A_\alpha : \alpha \in \Lambda\}$ *is a subset of* $\mathcal{T}$, *then* $\cup_{\alpha \in \Lambda} A_\alpha \in \mathcal{T}$, *where* Λ *is an index set.*

Each member of $\mathcal{T}$ is called a $\mathcal{T}$ **- open** (simply, an **open**) fuzzy set in X. F is called a $\mathcal{T}$ **-closed** (simply, a **closed**) fuzzy set in X, if $F^c \in \mathcal{T}$. The pair $(X, \mathcal{T})$ is called a **fuzzy topological space** (simply, **fts**).

For a fts X, we will denote the collection of all the open (resp. closed) fuzzy sets in X as $OF(X)$ (resp.$CF(X)$).

For a set X, the collection $\{\underline{0}_X, \underline{1}_X\}$ is called the indiscrete fuzzy topology on X, and I^X the discrete fuzzy topology on X.

For an infinite set,the collection $Coc_f(X) = \{A \in I^X : A^c \text{is countable}\} \cup \{\underline{0}_X\}$ is called the **cocountable fuzzy topology**) on X.

Definition 2.4[3]. *Let X be a fts and let $A \in I^X$. Then :*

(1) *The union of all the open fuzzy sets in A is called the* **interior** *of A, and denote by* $\mathring{A}$, $int_X A$ *or* $intA$. *Hence,*

$$\mathring{A} = \cup\{O \in OF(X) : O \subset A\}.$$

(2) *The intersection of all the closed fuzzy sets in A is called the* **closure** *of A, and denote by $\overline{A}, cl_X A$ or clA. Hence,*

$$\overline{A} = \cap\{F \in CF(X) : A \subset F\}.$$

It is clear that A is the largest open fuzzy set contained in A and $(\mathring{A})^{\circ} = A$; $\overline{A}$ is the smallest closed set containing A and $\overline{(\overline{A})} = \overline{A}$.

Definition 2.5[3]. *Let X be a fts, $A \in I^X$ and $x_\lambda \in F_p(X)$. Then A is called a* **neighborhood** *of x_λ, if there exists a $B \in OF(X)$ such that $x_\lambda \in B \subset A$; a neighborhood A is said to be* **open**, *if $A \in OF(X)$. The family of all the neighborhoods of x_λ is called the* **system of neighborhoods** *of x_λ and denoted by $\mathcal{N}_{x_\lambda}$, or $\mathcal{N}(x_\lambda)$.*

Definition 2.6[3]. *Let $(X, \mathcal{T})$ be a fts and let A be a subset of X. Then the family $\mathcal{T}_A = \{U|_A : U \in \mathcal{T}\}$ is called the* **subspace fuzzy topology** *on A to X. The pair $(A, \mathcal{T}_A)$ is called a* **fuzzy subspace** *of X.*

From Definition 2.6, we can easily obtain the following result:

Lemma 2.C. *Let $(A, \mathcal{T}_A)$ be a fuzzy subspace of a fts $(X, \mathcal{T})$. Then for each $V \in \mathcal{T}_A$, there exists $U \in \mathcal{T}$ such that $V = U \cap A$.*

Definition 2.7[4]. *Let X and Y be fts's, and let $x_\lambda \in F_p(X)$. Then a mapping $f : X \to Y$ is said to be* **fuzzy continuous** *(simply.* **F - continuous***) at x_λ if for each open neighborhood V of $f(x_\lambda)$, there exists an open neighborhood U of x_λ such that $f(U) \subset V$. And the mapping f is said to be fuzzy continuous (on X) if it is F - continuous at each $x_\lambda \in F_p(X)$.*

Lemma 2.D[4]. *Let X and Y be fts's and let $f : X \to Y$ be a mapping. Then the following statement are equivalent :*

(1) *f is F - continuous.*
(2) *For each $V \in OF(Y), f^{-1}(V) \in OF(X)$.*
(3) *For each $F \in CF(Y), f^{-1}(F) \in CF(X)$.*

Definition 2.8[2]. *A mapping $f : X \to Y$ is called a* **fuzzy homeomorphism** *(simply,* **F - homeomorphism***) if f is bijective, and f and f^{-1} are F - continuous. The two fts's are said to be* **F - homeomorphic** *if there exists an F - homeomorphism of one fts onto another. Such a case, the two fts's are said to be* **topologically F - equivalent**

3. Fuzzy Strongly Continuous Mappings.

Definition 3.1. *A mapping* $f : X \to Y$ *is said to be* **fuzzy strongly continuous** *if for each* $A \in I^X, f(clA) \subset f(A)$ *i.e.* $f(clA) = f(A)$.

Proposition 3.2. *Every fuzzy strongly continuous mapping is F - continuous. However the converse is not true, as shown by Examples 3.3 and 3.4 .*

Proof. Suppose a mapping $f : X \to Y$ is fuzzy strongly continuous and let $F \in CF(Y)$. Then, by the hypothesis, $f(clf^{-1}(F)) \subset f(f^{-1}(F)) \subset F$. Thus $clf^{-1}(F) \subset f^{-1}(F)$. So $clf^{-1}(F) = f^{-1}(F)$ and hence $f^{-1}(F) \in CF(X)$. Therefore, by Lemma 2.D, f is F-continuous. ■

Example 3.3. *Let* $Y = \{a, b, c\}$ *and let* $\mathcal{T}^* = \{\underline{0}_Y, O, \underline{1}_Y\}$, *where* $O = \{(a, 0), (b, 0), (c, \lambda)\}$ *and* $\lambda \in (0, 1]$. *Consider the mapping* $f : (R, Coc_f(R)) \to (Y, \mathcal{T}^*)$ *defined by for each* $x \in R$

$$f(x) = \begin{cases} a \text{ if } x \text{ is rational} \\ b \text{ if } x \text{ is irrational.} \end{cases}$$

Then f is F - continuous but not fuzzy strongly continuous.

Example 3.4. *Let X be a set and let* $\mathcal{T}$ *be the fuzzy indiscrete topology on X. Consider the identity mapping* $id_X : (X, \mathcal{T}) \to (X, \mathcal{T})$. *Then* id_X *is F-continuous but not fuzzy strongly continuous.*

Remark 3.5. *Example 3.4 provides also that even a F - homeomorphism may fail to be fuzzy strongly continuous.*

Theorem 3.6. *A mapping* $f : X \to Y$ *is fuzzy strongly continuous if and only if for each* $B \in I^Y$, $f^{-1}(B) \in CF(X) \cap OF(X)$.

Proof. $(\Rightarrow)$: Suppose f is fuzzy strongly continuous and let $B \in I^Y$ Then, by the hypothesis, $f(clf^{-1}(B)) \subset f(f^{-1}(B)) \subset B$. Thus $clf^{-1}(B) \subset f^{-1}(B)$, and hence $f^{-1}(B) \in CF(X)$. Also, by the hypothesis, $f(clf^{-1}(B^c) \subset f(f^{-1}(B^c)) \subset B^c$. Thus $clf^{-1}(B^c) \subset f^{-1}(B^c)$, and hence $f^{-1}(B^c) \in CF(X)$. But, by Lemma 2.A , $f^{-1}(B^c) = [f^{-1}(B)]^c$. So $[f^{-1}(B)]^c \in CF(X)$ and hence $f^{-1}(B) \in OF(X)$. Therefore $f^{-1}(B) \in CF(X) \cap OF(X)$.
$(\Leftarrow)$: Suppose $f^{-1}(B) \in CF(X) \cap OF(X)$ for each $B \in I^Y$. Let $A \in I^X$. Then clearly $f^{-1}(f(A)) \in CF(X)$. But $A \subset f^{-1}(f(A))$. So $clA \subset f^{-1}(f(A))$ and thus $f(clA) \subset f(A)$. Hence f is fuzzy strongly continuous. ■

Corollary 3.7. *A mapping $f : X \to Y$ is fuzzy strongly continuous if and only if $f^{-1}(y_\lambda) \in OF(X) \cap CF(X)$ for each $y_\lambda \in F_p(Y)$.*

Definition 3.8. *A mapping $f : X \to Y$ is said to be* **fuzzy weakly continuous** *if for each $x_\lambda \in F_p(X)$ and each $H \in OF(Y)$ containing $f(x_\lambda)$, there exists a $G \in OF(X)$ containing x_λ such that $f(G) \subset clH$.*

Lemma 3.A. *A mapping $f : X \to Y$ is fuzzy weakly continuous if and only if for each $V \in OF(Y), f^{-1}(V) \subset int f^{-1}(clV)$.*

Theorem 3.9. *Every Fuzzy weakly continuous mapping into a fuzzy discrete space is fuzzy strongly continuous.*

Proof. Let $f : X \to Y$ be fuzzy weakly continuous and let Y a fuzzy discrete space. Let $B \in I^Y$. Since Y is a fuzzy discrete space, $B \in CF(Y) \cap OF(Y)$. Since f is fuzzy weakly continuous, $f^{-1}(B) \subset int f^{-1}(clB)$. So $f^{-1}(B) \subset int f^{-1}(B)$, and hence $f^{-1}(B) \in OF(X)$. Similarly, we have $f^{-1}(B) \in CF(X)$. Therefore f is fuzzy strongly continuous. ∎

Theorem 3.10. *Restriction of a fuzzy strongly continuous mapping $f : X \to Y$ to any subset of X is fuzzy strongly continuous.*

Proof. Let A be any subset of X and let $y_\lambda \in F_p(Y)$. Since f is fuzzy strongly continuous, by Corollary 3.7 $f^{-1}(y_\lambda) \in CF(X) \cap OF(X)$. But $(f|A)^{-1}(y_\lambda) = f^{-1}(y_\lambda) \cap A$. So $(f|A)^{-1}(y_\lambda) \in CF(A) \cap OF(A)$. Hence $f|A$ is fuzzy strongly continuous. ∎

Theorem 3.11. *If $f : X \to Y$ is fuzzy strongly continuous and $g : Y \to Z$ any mapping, then $g \circ f : X \to Z$ is fuzzy strongly continuous.*

Proof. Let $A \in I^Z$. Then clearly $g^{-1}(A) \in I^Y$. Since f is fuzzy strongly continuous by theorem 3.6 $f^{-1}(g^{-1}(A)) \in CF(X) \cap OF(X)$. But $f^{-1}(g^{-1}(A)) = (g \circ f)^{-1}(A)$. So $(g \circ f)^{-1}(A) \in CF(X) \cap OF(X)$. Hence $g \circ f$ is fuzzy strongly continuous. ∎

Corollary 3.12. *The composite of two fuzzy strongly continuous mappings is fuzzy strongly continuous.*

The following example shows that the result of Theorem 3.11 is not necessarily true for F - continuous mappings:

Example 3.13. *Let X be any set and let $\mathcal{T}$ and $\mathcal{T}^*$ be the fuzzy indiscrete topology, and the fuzzy discrete topology on X, respectively, Consider the identity mapping $f : (X, \mathcal{T}) \to (X, \mathcal{T})$ and $g : (X, \mathcal{T}) \to (X, \mathcal{T}^*)$. Then clearly f is F - continuous. But $g \circ f$ is not F - continuous.*

Theorem 3. 14. *If $f : X \to Y$ is fuzzy weakly continuous and $g : Y \to Z$ is fuzzy strongly continuous, then $g \circ f$ is fuzzy strongly continuous.*

Proof. Let $A \in I^Z$. Since g is fuzzy strongly continuous, $g^{-1}(A) \in CF(Y) \cap OF(Y)$. Since f is fuzzy weakly continuous, by Lemma 3.A , $f^{-1}(g^{-1}(A)) \subset intf^{-1}(clg^{-1}(A)) = intf^{-1}(g^{-1}(A))$. So $f^{-1}(g^{-1}(A)) \in OF(X)$ and thus $(g \circ f)^{-1}(A) \in OF(X)$. Similary, by Lemma 3.A. we have $(g \circ f)^{-1}(A) \in CF(X)$. Hence $g \circ f$ is fuzzy strongly continuous. ■

Corollary 3.15. *If $f : X \to Y$ is F - continuous and $g : Y \to Z$ fuzzy strongly continuous, then $g \circ f : X \to Z$ is fuzzy strongly continuous.*

Corollary 3.16. *Let $f : X \to \Pi_{\alpha \in \Lambda} X_\alpha$ be fuzzy strongly continuous. For each $\alpha \in \Lambda$, let $f_\alpha : X \to X_\alpha$ be defined as $f_\alpha(x) = x_\alpha$ if $f(x) = (x_\alpha)$. Then f_α is fuzzy strongly continuous for each $\alpha \in \Lambda$*

Proof. Let π_α denoted the projection of $\Pi_{\alpha \in \Lambda} X_\alpha$ onto X_α. Then, obviously $f_\alpha = \pi_\alpha \circ f$ for each $\alpha \in \Lambda$ and π_α is F - continuous for each $\alpha \in \Lambda$. Therefore, by Corollary 3.15, f_α is fuzzy strongly continuous for each $\alpha \in \Lambda$. ■

Corollary 3.17. *Let $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$ be fuzzy strongly continuous mappings. Let $X = X_1 \times X_2$ and $Y = Y_1 \times Y_2$. Let $f : X \to Y$ be defined as $f(x_1, x_2) = (f(x_1), f(x_2))$. Then f is fuzzy strongly continuous.*

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