집합치 함수의 Choquet 적분에 관한 성질

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Some Properties of Choquet Integrals of Set-Valued Functions

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Abstract : In this paper, we will define Choquet integrals of set-valued functions. Then, we show some properties of Choquet integrals of set-valued functions.

Key words : Analysis, Generalized measure theory, Fuzzy measures, Set-valued functions, Choquet integrals, Set-valued Choquet integrals.

1. Introduction

The concept of set-valued functions have been used repeatedly in D. Zhang and Z.Wang[2] and Zhang Guangquan[4]. Fuzzy integrals of set-valued functions have been studied by D. Zhang and Z. Wang[2] and C.L. Byrne[3]. Sugeno[8] began the research of real-valued fuzzy measure and real-valued fuzzy integrals of real-valued functions with respect to a real-valued fuzzy measure on a classical set. But the integrands are all point-valued functions. And also, D.Zhang and Z. Wang[1,2] introduced the definition of fuzzy integrals of set-valued functions(simply, said to be set-valued fuzzy integrals) and proved some properties of set-valued fuzzy integrals. In section 2, we will define Choquet integrals of set-valued functions(simply, said to be set-valued Choquet integrals), which is a natural extension of that of point-valued functions. And we will list various definitions and notations which will be used in the sequel. In section 3 we will prove some properties of Choquet integrals of set-valued functions.

2. Definitions

In this section, we will define set-valued Choquet integrals. Then, it is a natural extension of Choquet integrals of point-valued functions. At first, we will introduce the following definitions and notations. R^+ will denote the interval $[0,\infty]$, and $C(R^+)$ denote the class of all closed subsets of R^+.

Definition 2.1([6],[7],[9]). *A fuzzy measure on a measurable space $(X, \mathcal{A})$ is a set function $\mu : X \longrightarrow [0,1]$ with a properties:*

(1) $\mu(\emptyset) = 0$,

(2) $\mu(X) = 1$,

(3) *if* $A \subset B$, *then* $\mu(A) \leq \mu(B)$.

We note that monotonicity and non-additivity are the main features of this measure.

The Choquet integral([6]) of a measurable real-valued function $f : X \longrightarrow R^+$ with respect to a fuzzy measure μ is defined by

$$(C)\int f d\mu = \int_0^{\infty} \mu\{\omega | f(\omega) > \alpha\}) d\alpha$$

where the integral of the right hand side is ordinary one. A set-valued function is a function from X to $\mathcal{C}(R^+) \setminus \{\emptyset\}$.

Definition 2.2. *A set-valued function F is said to be measurable if its graph is measurable, that is,*

$$Gr(F) = \{(x, r) \in X \times R^+ | r \in F(x)\} \in \mathcal{A} \times \mathcal{B}(R^+)$$

where $\mathcal{B}(R^+)$ is the Borel field of R^+.

Definition 2.3([10]). *Let $(X, \mathcal{A}, \mu)$ be a fuzzy measure space. A function $f : X \longrightarrow R^+$ is said to be measurable if $\{x | f(x) > \alpha\}$ is measurable for all $\alpha \in R^+$.*

We say $f : X \longrightarrow R^+$ is in $L^1(\mu)$ if and only if f is measurable and $(C)\int f d\mu < \infty$.

Definition 2.4. *Let F be a set-valued function and $A \in \mathcal{A}$. Then, a Choquet integral of F on $\mathcal{A}$ is defined by*

$$(C)\int F d\mu = \{(C)\int_A f d\mu | f \in S(F)\},$$

where $S(F) = \{f \in L^1(\mu) | f(x) \in F(x) \quad a.e.[\mu] \quad on \quad X\}$. That is, the family of a.e. $[\mu]$ Choquet integrable selection of F.

Instead of $(C)\int_X F d\mu$, we will write $(C)\int F d\mu$. Obviously, $(C)\int F d\mu$ may be empty.

Definition 2.5. *A set-valued function F is said to be Choquet integrably bounded if there is $g \in L^1(\mu)$ such that*

$$||F(x)|| = \sup_{r \in F(x)} |r| \leq g(x).$$

3. Properties of set-valued Choquet Integrals

From those definitions in section 2, we obtain the following propositions.

Proposition 3.1. *If F is a Choquet integrably bounded set-valued function, then F is Choquet integrable.*

Proof: By the definition 2.5, there is $g \in L^1(\mu)$ with $f(x) \leq g(x)$ a.e. $[\mu]$ for all $f \in S(F)$. Hence, we have

$$(C)\int f d\mu \leq (C)\int g d\mu < \infty.$$

So f is Choquet integrable and hence $(C)\int F d\mu \neq \emptyset$. Therefore, F is a Choquet integrable.

From the proposition 3.1, we can see that the Choquet integrability of point-valued functions is a special case of that of set-valued functions, since a point-valued function can be viewed as a special set-valued function.

Now we will show 'monotonicity' in the following. Let $A, B \in \mathcal{C}(R^+)$. Then, $A \leq B$ means that

(1) for each $x_0 \in A$, there exists $y_0 \in B$ such that $x_0 \leq y_0$,

(2) for each $y_0 \in B$, there exists $x_0 \in A$ such that $x_0 \leq y_0$.

Proposition 3.2. *Let F be a Choquet integrable set-valued function. And let $A, B \in \mathcal{A}$. Then $A \subset B$ implies*

$$(C)\int_A F d\mu \leq (C)\int_B F d\mu.$$

Proof: Let $x_0 \in (C)\int_A F d\mu$. Then, there exists $f \in S(F)$ such that

$$x_0 = (C)\int_A f d\mu = (C)\int f\chi_A d\mu.$$

Since $f \geq 0$ a.e. $[\mu]$ on X and $A \subset B$, $f\chi_A \leq f\chi_A$. By theorem 3.1 (d) of [5],

$$(C)\int f\chi_A d\mu \leq (C) f\chi_B d\mu.$$

Since $f \in S(F)$, $(C)\int f\chi_B d\mu = (C)\int_B f d\mu$ lies in $(C)\int_B F d\mu$. If we put $y_0 = (C)\int_B f d\mu$, then $x_0 \leq y_0$ and $y_0 \in (C)\int_B F d\mu$. Let $y_0 \in (C)\int_B F d\mu$. Then, there exists $f \in S(F)$ such that

$$y_0 = (C)\int_B f d\mu = (C)\int f\chi_B d\mu.$$

Since $f \geq 0$ a.e.$[\mu]$ on X and $A \subset B$, $f\chi_A \leq f\chi_B$. By theorem 3.1 (d) of [5],

$$y_0 = (C)\int f\chi_B d\mu \geq (C)\int f\chi_A d\mu.$$

Since $f \in S(F)$, $(C)\int f\chi_A d\mu = (C)\int_A f\mu$ lies in $(C)\int_A F d\mu$. If we put $x_0 = (C)\int_A f d\mu$, then $y_0 \geq x_0$ and $x_0 \in (C)\int_A F d\mu$. Therefore, $(C)\int_A F d\mu \leq (C)\int_B F d\mu$.

Proposition 3.3. *Let F_1 and F_2 be Choquet integrably bounded set-valued functions. If $F_1 \leq F_2$, then $(C)\int F_1 d\mu \leq (C)\int F_2 d\mu$.*

Proof: Let $x_0 \in (C)\int F_1 d\mu$. Then, there exists $f_0 \in S(F_1)$ such that $x_0 = (C)\int f_0 d\mu$. Since $F_1 \leq F_2$, $F_1(\omega) \leq F_2(\omega)$ for each $\omega \in X$. Define

$$g_0(\omega) = \sup\{x | x \in F_2(\omega) \text{ and } x \geq f_0(\omega)\}.$$

Now since $F_2(\omega)$ is closed, we have $g_0(\omega) \in F_2(\omega)$ and $g_0 \geq f_0(\omega)$. Since F_2 is Choquet integrably bounded, there exists $h \in L^1(\mu)$ such that $\sup_{r \in F_2(\omega)} |r| \leq h(\omega)$. Hence $g_0(\omega) \leq h(\omega)$ and $g_0 \in L^1(\mu)$. By theorem 3.1 (d) of [5], $x_0 = (C)\int f_0 d\mu \leq (C)\int g_0 d\mu = y_0$ and $y_0 = (C)\int g_0 d\mu \in (C)\int F_2 d\mu$.

Let $y_0 \in (C)\int F_2 d\mu$. Then, there exists $g_0 \in S(F_2)$ such that $f_0 = (C)\int g_0 d\mu$. Since $F_1 \leq F_2$, we have $F_1(\omega) \leq F_2(\omega)$ for each $\omega \in X$. Define

$$f_0(\omega) = \inf\{x | x \in F_1(\omega) \quad \text{and} \quad x \leq g_0(\omega)\}.$$

Since $F_1(\omega)$ is closed, we have $f_0(\omega)$ and $f_0(\omega) \leq g_0(\omega)$. By theorem 3.1 (d) of [5] again, $f_0 \in L^1(\mu)$ and $y_0 = (C)\int g_0 d\mu \geq (C)\int f_0 d\mu = x_0$. Hence $y_0 \geq x_0$ and $x_0 \in (C)\int F_1 d\mu$. Therefore $(C)\int F_1 d\mu \leq (C)\int F_2 d\mu$.

Proposition 3.4. *Let F be a Choquet integrable set-valued function. If $\mu(A) = 0$, then $(C)\int_A F d\mu = \{0\}$.*

Proof: Let $f \in S(F)$. Since a fuzzy measure μ has monotonicity and $S \cap \{x | f(x) > \alpha\} \subset A$ for each $\alpha \in R$, we have

$$\mu(A \cap \{x | f(x) > \alpha\}) \leq \mu(A) = 0, \quad \text{for each} \quad \alpha \in R.$$

Hence, we obtain

$$(C)\int_A f d\mu = \int_0^\infty \mu(A \cap \{x | f(x) > \alpha\}) d\alpha = 0.$$

By the definition of a set-valued Choquet integral of F, $(C)\int F d\mu = \{0\}$.

Proposition 3.5. *If F is a Choquet integrable a set-valued function, then $(C)\int aF d\mu = a(C)\int F d\mu$ for each $a > 0$.*

Proof: Let $y_0 \in (C)\int aF d\mu$. Then, there exists a function $g_0 \in S(aF)$ such that $y_0 = (C)\int g_0 d\mu$. We define $f_0(\omega) = \frac{1}{a} g_0(\omega)$. Since $g_0(\omega) \in (aF)(\omega) = aF(\omega)$, we have $f_0(\omega) \in F(\omega)$. Since $g_0 \in L^1(\mu)$, $f_0 = \frac{1}{a} g_0 \in L^1(\mu)$. Hence $f_0 \in S(F)$ and $a(C)\int f_0 d\mu \in a(C)\int F d\mu$. By proposition 3.2 (d) of [11], we have

$$y_0 = (C)\int g_0 d\mu = (C)\int a f_0 d\mu = a(C)\int f_0 d\mu \in a(C)\int F d\mu.$$

Thus, we have

$$(C)\int aFd\mu \subset a(C)\int Fd\mu. \tag{1}$$

Let $x_0 \in a(C)\int Fd\mu$. Then, $\frac{1}{a}x_0 \in (C)\int Fd\mu$. So, we have that there exists a function $f_0 \in S(F)$ such that $\frac{1}{a}x_0 = (C)\int f_0 d\mu$. Put $g_0 = af_0(\omega)$. Since $f_0(\omega) \in F(\omega)$ and $f_0 \in L^1(\mu)$, we have $g_0(\omega) = af_0(\omega) \in aF(\omega) = (aF)(\omega)$ and $g_0 = af_0 \in L^1(\mu)$. Hence $g_0 \in S(F)$ and $(C)\int af_0 d\mu$ lies in $(C)\int aFd\mu$. By proposition 3.2 (d) of [11] again,

$$x_0 = a(C)\int f_0 d\mu = (C)\int af_0 d\mu \in (C)\int aFd\mu.$$

So we have

$$a(C)\int Fd\mu \subset (C)\int aFd\mu. \tag{2}$$

From (1) and (2), we complete the proof.

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