INTERNAL GENERATION OF WAVES FOR TIME-DEPENDENT MILD-SLOPE EQUATIONS

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1 INTRODUCTION

As an alternative to the Berkhoff's (1972) elliptic equation approach to predict the transformation of waves, several coastal engineers obtained solutions to the time-dependent mild-slope equations which were derived by Smith and Sprinks (1975) (or mathematically equivalent, Radder and Dingemans (1985)), Nishimura et al. (1983) (or mathematically equivalent, Copeland (1985)), Kubo et al. (1992), and Lee (1994). The application of the time-dependent mild-slope equations are better than the elliptic mild-slope equation in terms of reducing disk storage and computational time.

For the time-dependent model, waves can be generated by specifying values of the water surface elevation or particle velocity, etc. as desired at the outside boundary at each time step. Waves can also be generated by adding the values with desired energy to the computed ones at the internal boundary at each time step. When the first way is used, problems may occur because the waves which arrive at the wavemaker boundary from inside the domain would be trapped and cause unwanted addition of wave energy. However, the second way solves such problems by permitting waves freely pass across the wavemaker boundary while the desired wave energy is generated at the boundary.

Here, we study the internal generation of waves for three typical time-dependent mild-slope equations, Copeland's, Radder and Dingemans', and Kubo et al.'s equations, and find that the velocity of disturbances caused by the incident wave can be obtained from the viewpoint of energy transport rather than the previously argued mass transport. First, geometric optics approach is used to get the velocity of wave energy for the models. Second, internal generation of waves are studied. Third, numerical simulation of the models is made with the technique of generating waves internally. Finally, summary and discussions are presented.

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2 ANALYSIS OF MODELS BY GEOMETRIC OPTICS APPROACH

Copeland's equations are given by

$$\frac{\partial \eta}{\partial t} + \frac{\bar{C}}{\bar{C}_g} \nabla \cdot Q = 0 \tag{2.1}$$

$$\frac{\partial Q}{\partial t} + \bar{C}\bar{C}_g \nabla \eta = 0 \tag{2.2}$$

where $\bar{C}$ and $\bar{C}_g$ are the phase speed and group velocity, respectively, of a wave with the carrier angular frequency $\bar{\omega}$ and wavenumber $\bar{k}$, η is the water surface elevation, and Q is a vertically integrated function of particle velocity. Elimination of Q from equations (2.1) and (2.2) yields

$$\frac{\partial^2 \eta}{\partial t^2} - \frac{\bar{C}}{\bar{C}_g} \nabla \cdot (\bar{C}\bar{C}_g \nabla \eta) = 0 \tag{2.3}$$

Radder and Dingemans' equations are given by

$$\frac{\partial \eta}{\partial t} = -\nabla \cdot (\frac{\bar{C}\bar{C}_g}{g} \nabla \tilde{\phi}) + \frac{(\bar{\omega}^2 - \bar{k}^2 \bar{C}\bar{C}_g)}{g} \tilde{\phi} \tag{2.4}$$

$$\frac{\partial \tilde{\phi}}{\partial t} = -g\eta \tag{2.5}$$

where $\tilde{\phi}$ is the velocity potential at mean water level. Elimination of η from equations (2.4) and (2.5) yields Smith and Sprinks' equation in terms of η given by

$$\frac{\partial^2 \eta}{\partial t^2} - \nabla \cdot (\bar{C}\bar{C}_g \nabla \eta) + (\bar{\omega}^2 - \bar{k}^2 \bar{C}\bar{C}_g)\eta = 0 \tag{2.6}$$

The Kubo et al.'s equation is given by

$$\nabla \cdot (\bar{C}\bar{C}_g \nabla \hat{\eta}) + \bar{k}^2 \bar{C}\bar{C}_g \hat{\eta} + i\nabla \cdot (\frac{\partial}{\partial \omega}(\bar{C}\bar{C}_g) \nabla \frac{\partial \hat{\eta}}{\partial t}) + i\frac{\partial}{\partial \omega}(\bar{k}^2 \bar{C}\bar{C}_g)\frac{\partial \hat{\eta}}{\partial t} = 0 \ . \tag{2.7}$$

where $\hat{\eta}$ is related to the water surface elevation by

$$\eta = \hat{\eta} e^{-i\bar{\omega}t}. \tag{2.8}$$

For the case of a constant water depth, the propagation of surface waves is treated from the geometric optics approach which leads to the ray approximation. The water surface elevation can be defined as

$$\eta = A(x,t)e^{i\psi} \tag{2.9}$$

where the wave amplitude A modulates in time and space and the phase function ψ has the following relation with the local wavenumber k and angular frequency ω as

$$k = \nabla\psi, \quad \omega = -\tfrac{\partial\psi}{\partial t} \tag{2.10}$$

Substitution of equation (2.9) into Copeland's equation (2.3) yields the eikonal equation and energy transport equation as

$$\frac{k}{\bar{k}} = \frac{\omega}{\bar{\omega}} \tag{2.11}$$

$$\frac{\partial A^2}{\partial t} + \bar{C}\frac{\bar{C}}{C} \cdot \nabla A^2 = 0 \tag{2.12}$$

where the velocity of wave energy is

$$C_e = \bar{C} \tag{2.13}$$

Substitution of equation (2.9) into Smith and Sprinks' equation (2.6) yields the eikonal equation and energy transport equation as

$$\frac{k}{\bar{k}} = \sqrt{1 + \frac{1}{\bar{n}}\left(\left(\frac{\omega}{\bar{\omega}}\right)^2 - 1\right)} \tag{2.14}$$

$$\frac{\partial A^2}{\partial t} + \bar{C}_g\frac{\bar{C}}{C} \cdot \nabla A^2 = 0 \tag{2.15}$$

where $\bar{n} = \bar{C}_g/\bar{C} = (1 + 2\bar{k}h/\sinh 2\bar{k}h)/2$ and the velocity of wave energy is

$$C_e = \bar{C}_g\frac{\bar{\omega}}{\omega}\sqrt{1 + \frac{1}{\bar{n}}\left(\left(\frac{\omega}{\bar{\omega}}\right)^2 - 1\right)} \tag{2.16}$$

The function $\hat{\eta}$ in Kubo et al.'s equation (2.7) can be written as

$$\hat{\eta} = A(x,t)e^{i(\psi + \bar{\omega}t)} \tag{2.17}$$

Substitution of equation (2.17) into equation (2.7) yields the eikonal equation and energy transport equation as

$$\frac{k}{\bar{k}} = \sqrt{1 + \frac{2\left(\frac{\omega}{\bar{\omega}} - 1\right)}{\bar{n} + \left(\frac{\omega}{\bar{\omega}} - 1\right)\frac{\bar{k}^2}{\bar{\omega}}\frac{\partial}{\partial\omega}(\bar{C}\bar{C}_g)}} \tag{2.18}$$

$$\frac{\partial A^2}{\partial t} + \bar{C}_g\frac{k}{\bar{k}}\frac{1 + \frac{1}{\bar{n}}\left(\frac{\omega}{\bar{\omega}} - 1\right)\frac{\bar{k}^2}{\bar{\omega}}\frac{\partial}{\partial\omega}(\bar{C}\bar{C}_g)}{1 - \frac{1}{2}\left(\left(\frac{k}{\bar{k}}\right)^2 - 1\right)\frac{\bar{k}^2}{\bar{\omega}}\frac{\partial}{\partial\omega}(\bar{C}\bar{C}_g)} \cdot \nabla A^2 = 0 \tag{2.19}$$

where the velocity of wave energy is

$$C_e = \bar{C}_g\left(1 + \frac{1}{\bar{n}}\left(\frac{\omega}{\bar{\omega}} - 1\right)\frac{\bar{k}^2}{\bar{\omega}}\frac{\partial}{\partial\omega}(\bar{C}\bar{C}_g)\right)^2\sqrt{1 + \frac{2\left(\frac{\omega}{\bar{\omega}} - 1\right)}{\bar{n} + \left(\frac{\omega}{\bar{\omega}} - 1\right)\frac{\bar{k}^2}{\bar{\omega}}\frac{\partial}{\partial\omega}(\bar{C}\bar{C}_g)}} \tag{2.20}$$

Terms of second or higher order are neglected in deriving the eikonal equations (2.11), (2.14), (2.18), which are used in getting the velocities of wave energy given by equations (2.13), (2.16), (2.20), respectively.

3 INTERNAL GENERATION OF WAVES

Larsen and Dancy (1983) generated waves internally at the line parallel to the y-axis in the Boussinesq equations (Peregrine, 1967) with the added water surface elevations of incident wave as

$$\eta^* = 2\eta^I \frac{\tilde{C}\Delta t}{\Delta x}\cos\theta \tag{3.1}$$

where η^I is the water surface elevation of incident wave, θ is the angle of wave direction from the x-axis, Δx is the grid spacing in the x-axis, Δt is the time step. They argued that the velocity of disturbances in the x-direction caused by the incident wave is $\tilde{C}\cos\theta$ from the viewpoint of mass transport. Likewise, Madsen and Larsen (1987) and Yoon et al. (1996) generated waves in Copeland's equations with the added water surface elevations given by equation (3.1).

Monochromatic waves are generated with the added water surface elevations given by equation (3.1) for the three time-dependent mild-slope equations. The methods of simulating the equations are explained in section 4. For Copeland's equations, the ratio of the resulted to the desired wave amplitudes is found to be one in whole water depth. However, for Radder and Dingemans' and Kubo et al.'s equations, the ratio is found to be $1/\bar{n}$ which is larger than one in deep and intermediate-depth waters (see Fig.1). This requires a viewpoint different from the mass transport in order to properly generate waves internally for any time-dependent model.

The time-dependent models predict the evolution of wave energy as well as the wave phase. When the water surface elevation of incident wave is added to the computed one, there occurs an evolution of wave energy which is expressed by the energy transport equation. So, the velocity of wave energy needs to be used as a velocity of disturbances caused by the incident wave, and the added water surface elevations would be

$$\eta^* = 2\eta^I \frac{C_e\Delta t}{\Delta x}\cos\theta \tag{3.2}$$

The velocity of wave energy C_e for Copeland's equations is equal to the phase speed $\bar{C}$ accidentally.

4 NUMERICAL VERIFICATIONS

The Copeland's, Radder and Dingemans', and Kubo et al.'s equations are numerically simulated in one dimension to generate waves with the added water surface elevations given by equation (3.2). Three cases of the ratio of the local to the carrier wave frequencies are tested ($\omega/\bar{\omega} = 0.8, 1.0, 1.2$).

Sponge layers with the thickness three times the local wavelength are placed at outside boundaries to minimize wave reflections from the boundaries (Lee and Pyun, 1995). Thus, equation (2.2) is modified as

$$\frac{\partial Q}{\partial t} + \bar{C}\bar{C}_g \nabla \eta + \omega D_s Q = 0, \tag{4.1}$$

equation (2.5) is modified as

$$\frac{\partial \tilde{\phi}}{\partial t} = -g\eta - \omega D_s \tilde{\phi} \tag{4.2}$$

and equation (2.7) is modified as

$$\nabla \cdot (\bar{C}\bar{C}_g \nabla \hat{\eta}) + \bar{k}^2 \bar{C}\bar{C}_g (1 + 2iD_s)\hat{\eta} + i\nabla \cdot (\frac{\partial}{\partial \omega}(\bar{C}\bar{C}_g)\nabla \frac{\partial \hat{\eta}}{\partial t}) + i\frac{\partial}{\partial \omega}(\bar{k}^2 \bar{C}\bar{C}_g)(1 + 2iD_s)\frac{\partial \hat{\eta}}{\partial t} = 0 \tag{4.3}$$

The damping coefficient D_s increases exponentially from zero at the starting point of the sponge layer to one at the end.

The modified Copeland's equations (2.1) and (4.1) are discretized by a leap-frog method in a staggered grid in time and space. The modified Radder and Dingemans' equations (2.4) and (4.2) are discretized by a fourth-order Adams-Moulton predictor-corrector method in time and a three-point symmetric formula in space. The modified Kubo et al.'s equation (4.3) is discretized in time and space by the Crank-Nicolson method. The values at the initial time step are set to be zero. Equation (3.2) is multiplied by $\tanh(0.5t/T)$ to generate waves gradually. At outside boundaries, perfect reflection is assumed, which causes negligible reflections inside the domain bacause the sponge layer significantly reduces the incoming wave energy. The Courant number is $Cr = C_e\Delta t/\Delta x = 0.1$, the grid spacing is $\Delta x = L/20$, and the time step Δt is determined from the previous two conditions. The interior domain and two sponge layers cover 8 and 6 local wavelenths, respectively, (281 grid points in total). Waves are generated at the mid-point of the domain.

In Fig. 2, the normalized water surface elevations in steady state for the three equations are plotted along the lines with $y = 0, 3, 6$ for $\omega/\bar{\omega} = 0.8, 1.0, 1.2$, respectively. The cases of deep water ($\bar{k}h = 2\pi$) are shown. The wave amplitudes are shown to decay to almost zero at outside boundaries. For Kubo et al.'s equation, the normalized wave ampliutdes are much larger than one at the point of wave generation. For all the three equations, waves are generated properly as desired, which proves that the use of the velocity of wave energy is the right way for the added water surface elevations.

5 CONCLUSIONS

The technique of generating waves internally is studied for three time-dependent mild-slope equations developed by Copeland, Radder and Dingemans, and Kubo et al. For Radder and Dingemans' and Kubo et al.'s equations, the desired energy of incident wave cannot be obtained from the previously argued viewpoint of mass transport. This viewpoint suggests the use of the phase speed for the velocity of disturbances caused by the incident wave. However, for all the three equations, the desired energy can be obtained from the viewpoint of energy tranport which suggests the use of the velocity of wave energy. This idea can be extended to any time-dependent model.

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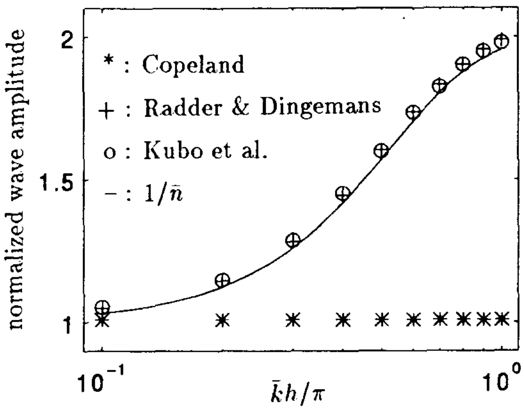

Fig. 1: Normalized wave amplitudes with $\eta^* = 2\eta^I \bar{C}\Delta t/\Delta x$

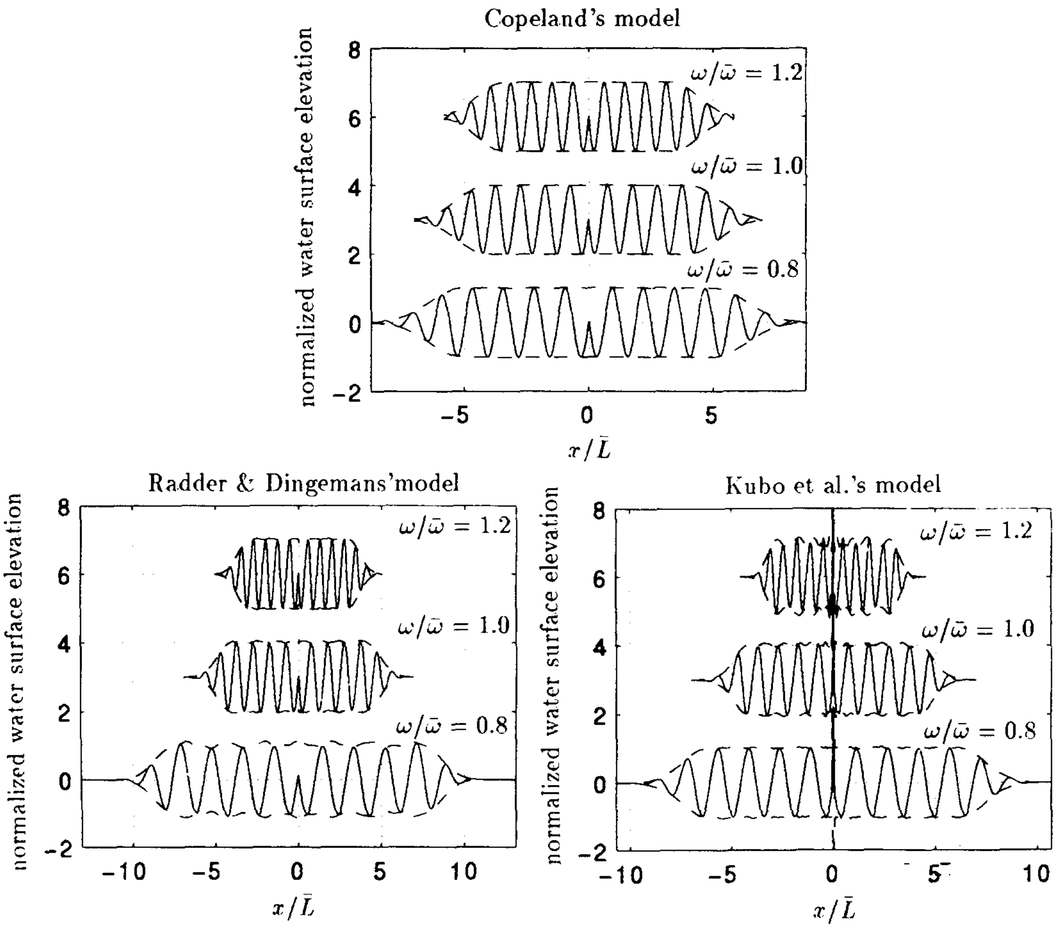

Fig. 2: Normalized water surface elevations with $\eta^* = 2\eta^I C_e \Delta t/\Delta x$ in deep water ($\bar{k}h = 2\pi$)