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정규모우드 방법을 활용한 진자형 흡진기의 비선형 동역학에 관한 연구

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On the Normal Mode Dynamics of a Pendulum Absorber

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Keywords : Normal mode, Bifurcation mode, Natural forcing function, Chaotic motion

ABSTRACT

By utilizing the concept of normal modes, nonlinear dynamics is studied on pendulum dynamic absorber. When the spring mode loses the stability in undamped free system, a dynamic two-well potential is formed in Poincaré map. A procedure is formulated to compute the forced responses associated with bifurcating mode and predict double saddle-loop phenomenon. It is found that quasiperiodic motion and stable periodic motion coexist in some parameter ranges, and only periodic motions or rotation of pendulum with chaotic fluctuation are observed in other ranges.

I. Introduction

Studies of the periodic motion at internal and external resonances in nonlinear coupled systems using perturbation techniques(i.e. multiple scale method, averaging method and etc.) have been progressed in actively.

As a representative example of these systems there be a pendulum dynamic absorber. One coordinate is directly excited by a harmonic force while the other is excited due to internal resonance. So it is called an autoparametric system.

Haxton and Barr[1] studied the system with limitations on the amplitude of harmonic excitation so that the primary response remains harmonic.

Hatwal, et al[2] investigated the forced vibration with the two types of restoring forces on the pendulum when a high levels of excitations applied at the internal resonance condition. Next they found the amplitude and phase modulated motion at the particular excitation amplitude and frequency conditions applying the harmonic balance method(HBM). Moreover, the existence of chaotic motion was observed at a large exciting amplitudes[3,4].

A.K. Bajaj, et al[5] studied the dynamics by the use of averaging method at the resonance condition in free and forced vibrations.

This study is investigated from the view point of normal modes, not resonance, to apply widely over a arbitrarily frequency ratio.

II. Basic Concept

Generally, there are some special periodic motions, known as the normal modes in equations of motion. At a sufficiently small system energy h, they are the linear normal modes and continuously extended to large

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systems energy. When h is very small, normal modes are always stable, but h is increased beyond a critical value(say bifurcation value), they may become unstable. By the stability change, a stable elliptic center (a) is replaced by a unstable saddle (b), as depicted in Fig. 1.

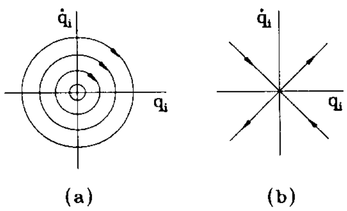

Fig. 1 Poincaré maps to show the stability changes of a periodic motion

It cab be shown that the stable and unstable manifolds of saddle create a saddle-loop or intersect each other because a hamiltonian system can not have a limit set. The former case is shown in Fig. 2(a), and in the latter case the intersection is generically transverse as shown in fig. 2(b) to have chaotic motion due to Smale-Birkhoff theorem[Guckenheimer and Holmes]. In either case there exists newly born periodic motion, i.e. bifurcation mode.

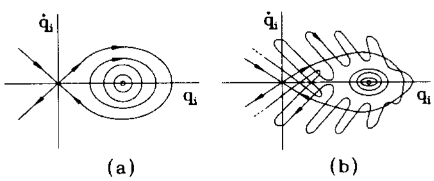

Fig. 2 The formation of a saddle-loop and a horse-shoe map

Also in forced vibration, it is predicted by the free vibration analysis that a quasiperiodic or chaotic approaching the saddle-loop and stable periodic motion coexist after the bifurcation in small damping. This is termed double saddle-loop phenomenon and cannot be found in perturbation analysis.

III. Equations of Motion

One coordinate x is excited by harmonic force. Primary mass M and secondary mass m exhibit translational and angular motion respectively. Restoring force of the pendulum is entirely due to gravity and rotational motion of m can be occur.

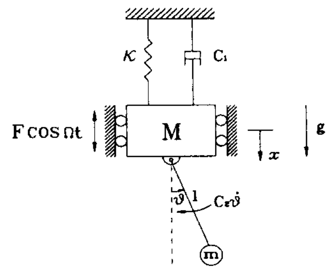

Fig. 3 The two d.o.f. autoparametric system

The kinetic energy T and potential energy V can be expressed in a dimensionless form

$$
\begin{aligned}
T &= \frac{1}{2}\dot{q}_1^{\,2} - \alpha \dot{q}_1 \dot{q}_2 \sin q_2 + \frac{1}{2}\alpha \dot{q}_2^{\,2} \\
V &= \frac{1}{2}\beta^2 q_1^2 - \alpha \cos q_2 \\
T + V &= h
\end{aligned} \quad (1)
$$

Where α is the mass ratio of the two bodies, β the ratio of two linear natural frequencies of the combined system. Then the equations of motion are written as

$$
\begin{aligned}
&\ddot{q}_1 + \beta^2 q_1 - \alpha \ddot{q}_2 \sin q_2 - \alpha \dot{q}_2^{\,2} \cos q_2 \\
&\quad + 2\sqrt{1-\alpha}\,\beta \zeta_1 \dot{q}_1 = F\cos \Omega t \\
&\ddot{q}_2 - \ddot{q}_1 \sin q_2 + \sin q_2 + 2\zeta_2 \dot{q}_2 = 0
\end{aligned} \quad (2)
$$

$$
\begin{aligned}
where,\ & q_1 = \frac{x}{l},\ q_2 = \theta,\ \alpha = \frac{m}{M+m},\ \beta = \frac{w_1}{w_2}, \\
& w_1 = \sqrt{\frac{k}{M+m}},\ w_2 = \sqrt{\frac{g}{l}}, \\
& \zeta_1 = \frac{C_1}{2\sqrt{Mk}},\ \zeta_2 = \frac{C_2}{2ml^2 w_2}
\end{aligned}
$$

IV. Free Vibration

The equations of motion are written as

$$
\begin{aligned}
\ddot{q}_1 + \beta^2 q_1 - \alpha \ddot{q}_2 \sin q_2 - \alpha \dot{q}_2^{\,2} \cos q_2 &= 0 \\
\ddot{q}_2 - \ddot{q}_1 \sin q_2 + \sin q_2 &= 0
\end{aligned} \quad (3)
$$

Where $H(q_1,q_2) \neq H(-q_1,q_2)$ so that symmetry property cannot persists about q_1.

There exists one trivial solution $q_2=0$. This motion $q_1 \neq 0$, $q_2=0$ is the spring mode or q_1 mode. It is straight line in the q_1-q_2 configuration space and satisfying all the properties of normal mode, hence called similar normal mode. There exists pendulum mode, $q_1=0$, $q_2 \neq 0$, in the linearized system but in case of regarding the nonlinear terms it is no longer pendulum mode. From the perturbation analysis, it is revealed that there are other types of periodic motions ; Nonlinear normal mode(NNM), Elliptic orbit(EO) and Bifurcation mode.

(1) Stability analysis of the spring mode

The stability of this normal mode is determined due to Synge's kinematico-statical sense[6] by the disturbance $q_2=\eta$, then

$$q_1(t)=A\cos\beta t\ ,\quad q_2(t)=0+\eta(t) \qquad (4)$$

Substitute Eqs. (4) into Eqs. (3) and linearize to obtain

$$\frac{d^2\eta}{d\tau^2}+(\delta+2\varepsilon\cos 2\tau)\eta=0 \qquad (5)$$
$$where,\quad \delta=\frac{4}{\beta^2},\ \varepsilon=2A,\ 2\tau=\beta t$$

The stability chart is shown in Fig. 4.

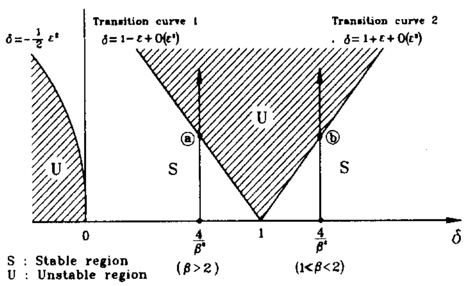

Fig. 4 Stability chart for the spring mode

Arrowed lines indicate the increase of energy and these cross the transition curve 1 or 2 with the two linear natural frequencies ratio β of the combined system. When the mode is stable, it is represented by an elliptic center, but it is replaced by a saddle when unstable. Two saddle-loops are formed, and a stable bifurcation mode are born in each loop ; a pitch-fork bifurcation[Fig. 5]. The eigenfunction corresponding to these transition curves are written as

$$\vec{\eta_1}(t)=c\left(\cos\frac{\beta t}{2}+\frac{1}{8}\varepsilon\cos\frac{3\beta t}{2}\right)+O(\varepsilon^2)$$
$$\vec{\eta_2}(t)=c\left(\sin\frac{\beta t}{2}+\frac{1}{8}\varepsilon\sin\frac{3\beta t}{2}\right)+O(\varepsilon^2) \qquad (6)$$

Where c is an arbitrarily chosen small constant.

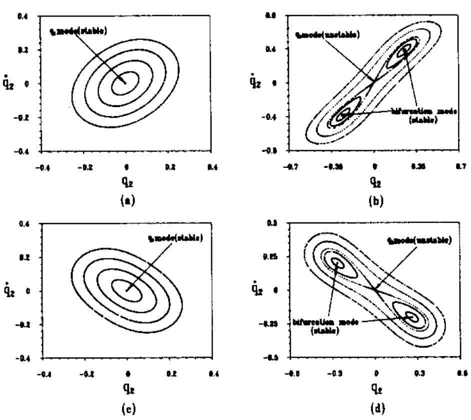

Fig. 5 Poincaré maps of the spring mode at before and after bifurcation
(a), (b) : $\alpha=0.1$, $\beta=2.5$
(c), (d) : $\alpha=0.1$, $\beta=1.8$

(2) Bifurcation mode

HBM can be used to calculate the bifurcation mode. It is expected that the solution of bifurcating mode can be calculated by assuming that the eigenfunction on transition curve. One term approximation is enough to depict the behavior of solution in accurately, so the generating functions for harmonic balance may be chosen in the form.

$$\beta>2,\qquad q_1(t)=A\cos wt,\quad q_2(t)=B\cos\frac{wt}{2}$$
$$1\ll\beta<2,\quad q_1(t)=A\cos wt,\quad q_2(t)=B\sin\frac{wt}{2} \qquad (7)$$

To apply HBM let $\sin q_2 \approx q_2$, $\cos q_2 \approx 1-q_2^2/2!$ from the kinetic energy T and potential energy V and applying Lagrange eq. then

$$\ddot{q}_1+\beta^2 q_1-\alpha\ddot{q}_2 q_2-\alpha\dot{q}_2^{\,2}=0$$
$$\ddot{q}_2-\ddot{q}_1 q_2+q_2=0 \qquad (8)$$

Substitute Eqs. (7) into Eqs. (8) and balancing harmonics

$$\beta>2, \quad \begin{aligned} &A(\beta^2-w^2)+\frac{1}{4}\alpha w^2B^2=0 \\ &B(1+\frac{1}{4}(2A-1)w^2)=0 \end{aligned} \tag{9a}$$

$$1\ll\beta<2, \quad \begin{aligned} &A(\beta^2-w^2)-\frac{1}{4}\alpha w^2B^2=0 \\ &B(1-\frac{1}{4}(2A+1)w^2)=0 \end{aligned} \tag{9b}$$

where, $B \neq 0$

Fig. 6 is the relation of A and B about ω and it is called the backbone curves of bifurcation mode. In some frequencies range bifurcation mode does not exist.

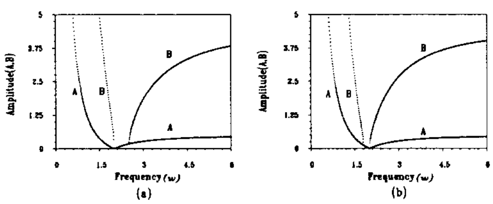

Fig. 6 Backbone curves of bifurcation mode
(a) : α=0.1, β=2.5, (b) : α=0.1, β=1.8

V. Forced Vibration

(1) Undamped forced vibration

In undamped forced vibration, i.e. $\zeta_1=\zeta_2=0$, equations of motion are expressed as following

$$\begin{aligned} &\ddot{q}_1+\beta^2q_1-\alpha\ddot{q}_2\sin q_2-\alpha\dot{q}_2^{\,2}\cos q_2=F\cos\Omega t \\ &\ddot{q}_2-\ddot{q}_1\sin q_2+\sin q_2=0 \end{aligned} \tag{10}$$

The concept of natural forcing function is used for the analysis of forced vibration which will produce the same form of solution as the free oscillations and its period is same that of forcing function[7].

① Stability analysis of the spring mode

To investigate the stability of the spring mode, η is perturbed in q_2 using the Synge's kinematico-statical sense.

$$q_1(t)=\frac{F}{\beta^2-\Omega^2}\cos\Omega t, \quad q_2(t)=0+\eta(t) \tag{11}$$

Substitute Eqs. (11) into Eqs. (10) and linearize to obtain

$$\eta''+(\delta+2\varepsilon\cos 2\tau)\eta=0 \tag{12}$$

$$\textit{where,}\ \delta=\frac{4}{\Omega^2},\ \varepsilon=\frac{2F}{\beta^2-\Omega^2},\ '=\frac{d}{d\tau}$$

The stability chart corresponding to Eq. (12) is Fig. 4 and the stability changes at the point ⓐ and ⓑ by increasing of control parameter F. Approximated bifurcation values $F_ⓐ$ and $F_ⓑ$ can be calculated by the transition curve eq.

$$\begin{aligned} &\beta>2, \quad F_ⓐ=\frac{(\Omega^2-4)(\beta^2-\Omega^2)}{2\Omega^2} \\ &1\ll\beta<2, \quad F_ⓑ=\frac{(4-\Omega^2)(\beta^2-\Omega^2)}{2\Omega^2} \end{aligned} \tag{13}$$

② Bifurcation mode

To find the bifurcation mode, the generating functions assumed as Eqs. (7) in which ω is replaced with Ω and substitute into assumed eq. of motion, of course it including the forcing term, and collecting the coefficients of harmonic terms, then the following equations derived.

$$\beta>2, \quad \begin{aligned} &A(\beta^2-\Omega^2)+\frac{1}{4}\alpha\Omega^2B^2=F \\ &B(1+\frac{1}{4}(2A-1)\Omega^2)=0 \end{aligned} \tag{14a}$$

$$1\ll\beta<2, \quad \begin{aligned} &A(\beta^2-\Omega^2)-\frac{1}{4}\alpha\Omega^2B^2=F \\ &B(1-\frac{1}{4}(2A+1)\Omega^2)=0 \end{aligned} \tag{14b}$$

If B=0, this motion is pure spring mode before bifurcation. When the excitation frequency Ω equals to the natural frequency ratio β of the combined system resonance occurs. Coupled motion occurs at $B \neq 0$ and Fig. 7 is the frequency response curves(FRC) of bifurcation mode at $F=F_ⓐ$ at Ω=2.28 and $F=F_ⓑ$ at Ω=1.94. When the forcing amplitude is small, FRC almost the same as backbone curves.

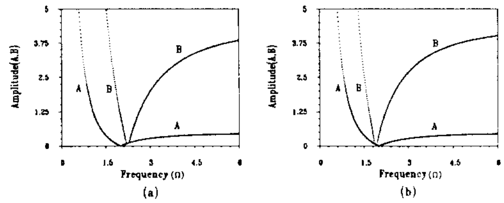

Fig. 7 Frequency response curves
(a) : α=0.1, β=2.5, (b) : α=0.1, β=1.8

③ Numerical simulation

To observe the existence of newly born periodic motion due to the stability change Eqs. (10) is numerically integrated by the 4th order Runge-kutta method. The parameters are α=0.1, β=2.5 and Ω=2.28 when the spring mode crosses the transition curve 1. In case of transition curve 2, α=0.1, β=1.8 and Ω=1.94. As expected, stable bifurcating modes observed and the period of spring motion is a half that of pendulum motion[Fig. 8, 9]. But as the forcing amplitude increases quasiperiodic motion appears instead of exact periodic motion result from HBM. Major cause lies in the approximation of eq. of motion in 2nd order to apply HBM.

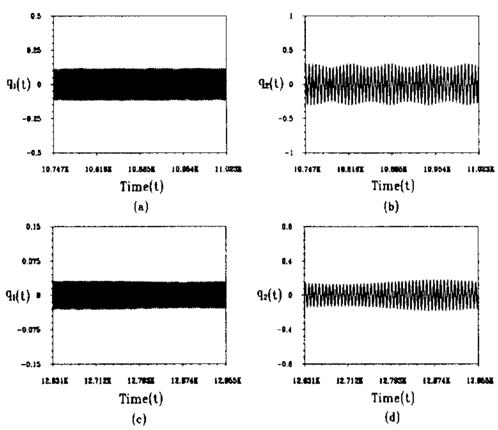

Fig. 8 Time responses of the undamped forced response associated with bifurcation mode
(a), (b) : β=2.5, F_w=1.1
(c), (d) : β=1.8, F_w=1.1

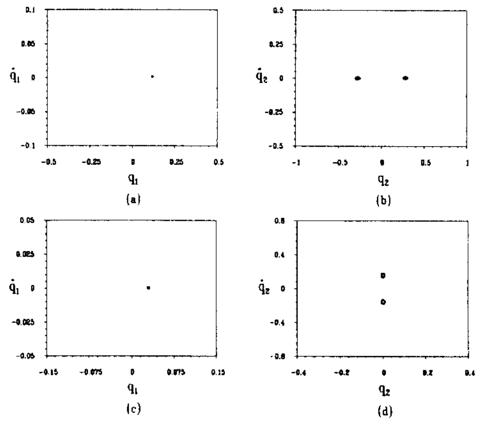

Fig. 9 Poincaré maps of the undamped forced response associated with bifurcation mode
(a), (b) : β=2.5, F_w=1.1
(c), (d) : β=1.8, F_w=1.1

And now displayed the representative behaviors of the coupled motion with the forcing weight F_{ω}(=F/$F_{ⓐ}$ or $F_{ⓑ}$) after the bifurcation. Chaotic motions including rotation of pendulum observed through the motions as shown in Fig. 10 and 11 with the increasing of forcing amplitude F. At β=2.5, energy transfer between the two motions is little but in case of β=1.8, it is much greater than β=2.5 and showing beating type motion.

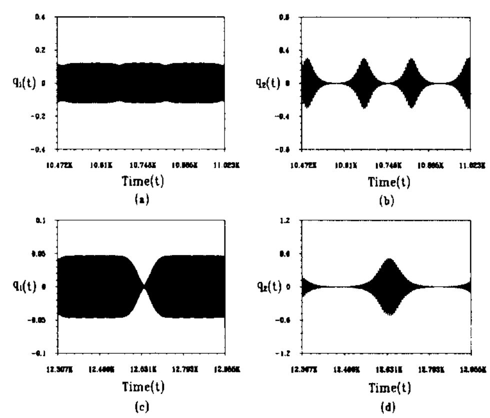

Fig. 10 Time responses of the coupled motion
(a), (b) : β=2.5, F_{ω}=1.05
(c), (d) : β=1.8, F_{ω}=1.50

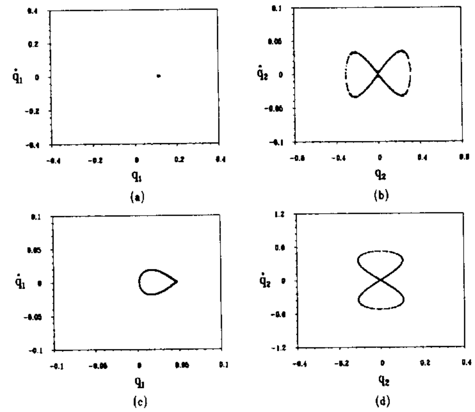

Fig. 11 Poincaré maps of the coupled motion
(a), (b) : β=2.5, F_{ω}=1.05
(c), (d) : β=1.8, F_{ω}=1.50

(2) Damped foced vibration

To apply HBM for forced responses associated with bifurcating mode Eqs. (2) assumed as following

$$\ddot{q}_1+\beta^2 q_1-\alpha\ddot{q}_2 q_2-\alpha\dot{q}_2^{\,2}+2\sqrt{1-\alpha}\beta\zeta_1\dot{q}_1 = F\cos\Omega t \qquad (15)$$

$$\ddot{q}_2-\ddot{q}_1 q_2+q_2+2\zeta_2\dot{q}_2=0$$

Where ζ_1 and ζ_2 are viscous damping factors.
The solutions are assumed in the form with the frequency ratio β.

$$\beta>2, \quad q_1(t)=A\cos(\Omega t-\phi_1), \quad q_2(t)=B\cos\left(\frac{\Omega t}{2}-\phi_2\right) \qquad (16a)$$

$$1\ll\beta<2, \quad q_1(t)=A\cos(\Omega t-\phi_1), \quad q_2(t)=B\sin\left(\frac{\Omega t}{2}-\phi_2\right) \qquad (16b)$$

It is observed that the following two cases $\beta>2$ and $1\ll\beta<2$ are through the same procedures. So only the $\beta>2$ case was investigated for forced responses.
Substitute Eqs. (16a) into Eqs. (15) to obtain

$$\begin{aligned}
&A\{(\beta^2-\Omega^2)\cos\phi_1+2\sqrt{1-\alpha}\zeta_1\beta\Omega\sin\phi_1\} + \frac{\alpha B^2\Omega^2}{4}\cos 2\phi_2=F\\
&A\{(\beta^2-\Omega^2)\sin\phi_1-2\sqrt{1-\alpha}\zeta_1\beta\Omega\cos\phi_1\} + \frac{\alpha B^2\Omega^2}{4}\sin 2\phi_2=0\\
&B\left\{-\frac{1}{4}\Omega^2\cos\phi_2+\frac{A\Omega^2}{2}\cos(\phi_1-\phi_2) + \cos\phi_2+\zeta_2\Omega\sin\phi_2\right\}=0\\
&B\left\{-\frac{1}{4}\Omega^2\sin\phi_2+\frac{A\Omega^2}{2}\sin(\phi_1-\phi_2) + \sin\phi_2-\zeta_2\Omega\cos\phi_2\right\}=0
\end{aligned} \qquad (17)$$

B=0 representing the pure spring mode response written as

$$q_1(t)=A\cos(\Omega t-\phi_1) \qquad (18)$$

where,

$$A=\frac{F}{\sqrt{(\beta^2-\Omega^2)^2+4(1-\alpha)\zeta_1^2\beta^2\Omega^2}}$$

$$\tan\phi_1=\frac{2\sqrt{1-\alpha}\beta\zeta_1\Omega}{\beta^2-\Omega^2}$$

When B≠0, the solution is associated with bifurcation mode and solved by using MATLAB as shown in Fig. 12. Also to verify the periodicity of responses time responses, Poincaré map and configuration space are computed as shown in Fig. 13. The parameters are same as shown in Fig. 12. It is readily seen that q_1 and q_2 behave periodic motion.

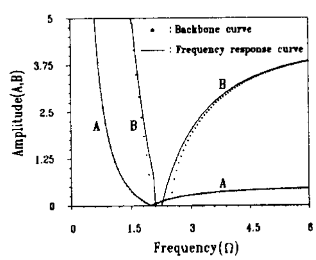

Fig. 12 Frequency response curves
α=0.1, β=2.5, F=F@ at Ω=2.28, $\zeta_1=\zeta_2=10^{-4}$

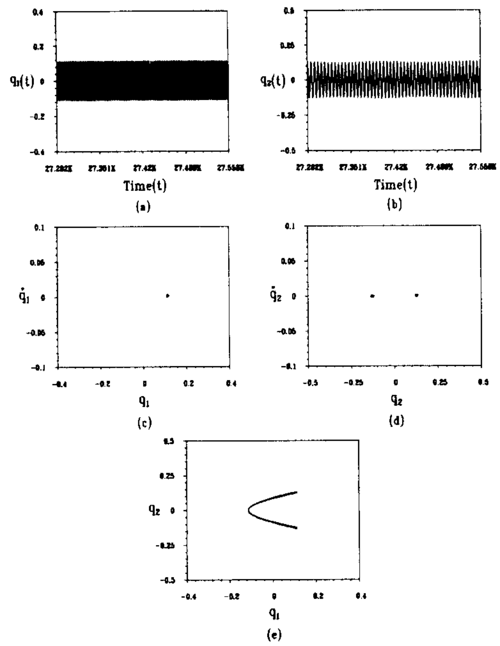

Fig. 13 The forced responses of bifurcation mode (a), (b) : time responses (c), (d) : Poincaré maps (e) : configuration space
$\alpha=0.1$, $\beta=2.5$, $F=F_{③}$ at $\Omega=2.28$, $\zeta_1=\zeta_2=10^{-4}$

For small damping and forcing amplitudes after the bifurcation, two-well phenomenon was found with the initial condition near spring mode. In Fig. 14, It is shown that the coexistence of quasiperiodic motion approaching the strange attractor and stable periodic motion.

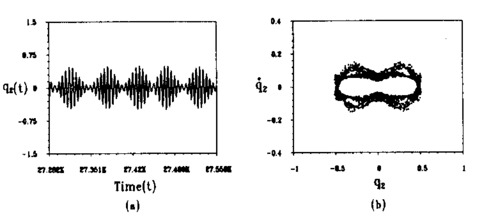

Fig. 14 Two-well phenomenon
$\alpha=0.1$, $\beta=2.5$, $F_w=1.14$, $\zeta_1=\zeta_2=10^{-4}$

Although it is not shown but in some parameter ranges strange attractor and periodic motion will be coexist from the series of bifurcation.

VI. Conclusion

(1) Generally in symmetric systems, saddle-loop is formed in poincaré map after the stability changes of normal mode. Also in the nonsymmetric system, above phenomenon was found. Two saddle-loops originating two-well phenomenon, and elliptic center is each loop originating limit cycles.
(2) It is verified by simulation that HBM is the useful tool to calculate the bifurcation mode.
(3) The two attracting limit cycles and quasiperiodic motions are represent the double saddle-loop phenomena. This come from the pitch-fork bifurcation of spring mode in the free vibration.
(4) A procedure is formulated to compute a bifurcating mode and its associated forced responses.

References

(1) R. S. Haxton and A. D. S. Barr, "The Autoparametric Vibration Absorber," ASME J. Eng. Ind., Vol. 94, 1972, pp. 119-125.
(2) H. Hatwal, A. K. Mallik, and A. Ghosh, "Nonlinear Oscillations of an Harmonically Excited Autoparametric System," J. Sound and Vibration, Vol. 81, No. 2, pp. 153-164.
(3) H. Hatwal, A. K. Mallik, and A. Ghosh, "Forced Nonlinear Oscillations of an Autoparametric System-Part 1: Periodic Responses," ASME J. Appl. Mech., Vol. 50, 1983, pp. 657-662.
(4) H. Hatwal, A. K. Mallik, and A. Ghosh, "Forced Nonlinear Oscillations of an Autoparametric System-Part 2: Chaotic Responses," ASME J. Appl. Mech., Vol. 50, 1983, pp. 663-668.
(5) A. K. Bajaj, S. I. Chang, and J. M. Johnson, "Amplitude Modulated Dynamics of a Resonantly Excited Autoparametric Two Degree-of-Freedom System," Nonlinear Dynamics 5, 1994, pp. 433-457.
(6) J. L. Synge, "On the Geometry of Dynamics", Phil. Traus 226, pp. 31-106.
(7) T. J. Harvey and P. Alto, "Natural Forcing Functions in Nonlinear Systems," J. Appl. Mech., 1958, pp. 352-356.