An Expansion Technique for Tolerance Approach to Sensitivity Analysis in Linear Programming

Koonchan Kim* · Young Soo Jo** · Kang Young Yug***
Department of Mathematics, Keimyung University, Daegu, Korea 704-701

Abstract

The *tolerance* approach to the sensitivity analysis in linear programming considers simultaneous and independent variations in the coefficients of the objective function or of the right-hand side terms and gives a region in which the coefficients and terms can be changed and still the current optimal basis **B** for the original problem remains as an optimal basis for the perturbed problem. In this paper we describe a procedure that expands a region S obtained by the tolerance approach into a larger region R, so that more variations in the objective function coefficients or the right-hand side terms are permissible.

1 Introduction

Suppose that the following linear programming has been solved and that **B** is the associated optimal basis produced by the simplex method.

$$\text{minimize } c^T x, \text{ s.t. } Ax = b, \ x \geq 0. \tag{1}$$

Suppose that one alters the coefficients of the objective function c or the right-hand side terms b in (1). A question is how much can one change them so that the current optimal basis remains optimal. Usually, the "ordinary" sensitivity analysis deals perturbations with one coefficient or one term at a time[1].

The tolerance approach[4] considers simultaneous and independent variations in the coefficients of the objective function or of the right-hand side terms and produces a region S within which the coefficients or right-hand side terms can be changed and still keeps the same optimal basis intact.

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The purpose of this paper is to describe a technique for expanding the region S into a larger region R such that $S \subseteq R$.

2 An Expansion Technique

Here we pose the following problem: Given a polytope P defined by the following set of inequalities,

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i \quad \text{for } i = 1, \ldots, m$$

with at least one interior point y° in P and $m > n$, we would like to find a specific region R with $y^\circ \in R$ as large as possible such that $R \subseteq P$. An approach would consist of finding an S contained in P and then we expand S as much as possible into R; of course, y° must be in S. Note that since y° is an interior point, $Ay^\circ < b$.

2.1 Computing a Region S

By denoting the above polytope as $Ax \leq b$, we want to find an α such that $A(y^\circ + \alpha) \leq b$. This can be written as $A\alpha \leq b - Ay^\circ = \tilde{b}$, i.e.,

$$\sum_{j=1}^{n} a_{ij} \alpha_j \leq \tilde{b}_i \quad \text{for } i = 1, \ldots, m \tag{2}$$

Note that $\tilde{b}_i$ is the ith component of $\tilde{b}$ and each $\tilde{b}_i > 0$ since y° is an interior point. To find a suitable α, we compute $\alpha_* \geq 0$ as the largest value such that so long as $-\alpha_* \leq \alpha_j \leq \alpha_*$, the inequalities in (2) should be satisfied. For any $\tilde{\alpha} \geq 0$, the largest that the left-hand side in (2) can ever get subject to $-\tilde{\alpha} \leq \alpha_j \leq \tilde{\alpha}$, $j = 1, \ldots, n$ is $(\sum_{j=1}^{n} \mid a_{ij} \mid)\tilde{\alpha}$ for each i. Hence, $\alpha_* = \max\{\tilde{\alpha} \geq$

$0 : \sum_{j=1}^{n} \mid a_{ij} \mid \tilde{\alpha} \leq \tilde{b}_i\}$ for each $i = 1, \ldots, m$. If we let $\beta_i = \frac{\tilde{b}_i}{\sum_{j=1}^{n} |a_{ij}|}$ for $i = 1, \ldots, m$, then

$$\alpha_* = \min\{\beta_1, \beta_2, \ldots, \beta_m\}. \qquad (3)$$

This computation is done based on the Tcheby-cheff norm, and see [4] for its application on the development of the tolerence approach. If we let each component of α be α_*, i.e., $\alpha = (\alpha_*, \alpha_*, \ldots, \alpha_*)^T$, then $A\alpha \leq \tilde{b}$ and

$$S = \prod_{j=1}^{n} [y_j^\circ - \alpha_*, y_j^\circ + \alpha_*] \qquad (4)$$

will be contained in P with $y^\circ \in S$. To illustrate this point, consider the following example whose graph is given in Figure 1.

$$\text{Example 1:} \quad \begin{aligned} -2x_1 + x_2 &\leq 2 \\ x_1 + x_2 &\leq 3 \\ 4x_1 - 5x_2 &\leq 20 \\ -x_2 &\leq 2 \end{aligned} \qquad (5)$$

First, we let $y^\circ = (2, 0)$. Then, $A\alpha \leq b - Ay^\circ = \tilde{b}$ becomes

$$\begin{aligned} -2\alpha_1 + \alpha_2 &\leq 6 \\ \alpha_1 + \alpha_2 &\leq 1 \\ 4\alpha_1 - 5\alpha_2 &\leq 12 \\ -\alpha_2 &\leq 2 \end{aligned} \qquad (6)$$

and $\alpha_*^\circ = \alpha_* = \min\{\ \beta_1 = \frac{6}{3}, \beta_2 = \frac{1}{2}, \beta_3 = \frac{12}{9}, \beta_4 = \frac{2}{1}\} = \frac{1}{2}$. The range of $S = [1.5, 2.5] \times [-0.5, 0.5]$ and its region is given in Figure 1, the dotted square.

2.2 Expansion of Region S

Next, we discuss an expansion R of S such that $S \subseteq R \subseteq P$. The expansion can be done as many times as possible by repeating the above steps. Upon obtaining α_*°, we suppose that an ℓth inequality in (6) becomes tight(if more than one, choose arbitrarily), i.e.,

$$\mid a_{\ell 1} \mid \alpha_*^\circ + \mid a_{\ell 2} \mid \alpha_*^\circ + \cdots + \mid a_{\ell n} \mid \alpha_*^\circ \ = \tilde{b}_\ell \quad (7)$$

We will find a point $z = (z_1, z_2, \ldots, z_n)$ belonging to the boundary of the ℓth inequality and which becomes a corner point(vertex) of S as follows.

Proposition 1 *If α_*° and S are obtained by (3) and (4), respectively, and ℓth inequality in (6) becomes tight, then a point $z = (z_1, z_2, \ldots, z_n)$ belonging to the boundary of the ℓth inequality of (5) and which becomes a corner point(vertex) of S is given by*

$$z_j = \begin{cases} y_j^\circ + \alpha_*^\circ & \text{if } a_{\ell j} \geq 0 \\ y_j^\circ - \alpha_*^\circ & \text{if } a_{\ell j} < 0, \end{cases} \quad j = 1, \ldots, n$$

In Example 1, for $y^\circ = (2, 0)$, we have $\ell = 2$ and $\alpha_*^\circ = \frac{1}{2}$. Since $a_{21} \geq 0$ and $a_{22} \geq 0$, we add $\frac{1}{2}$ both to y_1° and y_2° to obtain $z = (2.5, 0.5)$. This point belongs to the boundary of the second inequality of Example 1 (see Figure 1). Then we consider a direction eminating from the point z and passing through the point y°, and determine a point y^1 in P as follows:

$$y^1 = y^\circ + (y^\circ - z)$$

Note that the distance between the points z and y°, y° and y^1 are the same. In example 1, $y^1 = (1.5, -0.5)$ (see Figure 1).

Now, we repeat the procedure given in section 2.1 by starting at y^1, i.e., we replace y° by y^1. For $y^1 = (1.5, -0.5)$, $\tilde{b} = (5.5, 2.0, 11.5, 1.5)$. Then we calculate α_*^1 as in (3). At this point one of the two possible cases can occur:

$$(I) \quad \alpha_*^1 < 2\alpha_*^\circ$$
$$(II) \quad \alpha_*^1 = 2\alpha_*^\circ$$

If (I) occurs, then no expansion can be done and we must be satisfied with the region S. However, if (II) results, the region S can be expanded into a larger region R that contains S. For the above problem, $\alpha_*^1 = \min\{\ \beta_1 = \frac{5.5}{3}, \beta_2 = \frac{2}{2}, \beta_3 = \frac{11.5}{9}, \beta_4 = \frac{1.5}{1}\} = 1$. Note that $\alpha_*^1 = 2\alpha_*^\circ$ and hence it satisfies the second case (II). In this case the same z is obtained. By computing $y^2 = y^1 + (y^1 - z) = (0.5, -1.5)$, an expanded region R can be obtained as

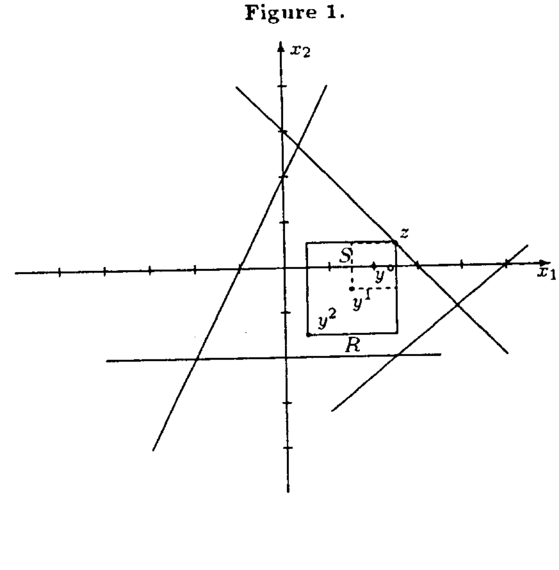
$$R = [0.5, 2.5] \times [-1.5, 0.5]$$

This is from

$$R = \prod_{j=1}^{n} [min\{z_j, y_j^i\}, max\{z_j, y_j^i\}] \qquad (8)$$

with $i = 2$. See Figure 1 below for the expansion from S to R.

Figure 1.



At y^2, one can further test for a possible expansion, but it can be easily shown that $\alpha_*^2 < 2\alpha_*^1$. Hence, no further expansion can be done for this problem.

2.3 An Expansion Algorithm

In this section the previous procedure is summarized as a tolerance approach expansion algorithm.

Algorithm TAE

begin
  Given a polytope $P(Ax \leq b)$, an interior point y°, $NMAX$
  STEP 0: $i \leftarrow 0$
    Form $A\alpha \leq b + Ay^\circ = f(y^\circ)$
    Compute α_*°, S at y°, and z.
    as in (3),(4) and by Proposition 1
  STEP 1: $i \leftarrow i+1$
    $R \leftarrow S$
    $y^i = y^{i-1} + (y^{i-1} - z)$
    Form $A\alpha \leq b + Ay^i = f(y^i)$
    Compute α_*^i as in (3)
  STEP 2: If $\alpha_*^i < 2\alpha_*^{i-1}$ or $i = NMAX$,
    exit with R
    If $\alpha_*^i = 2\alpha_*^{i-1}$ **then**
      Compute S at y^i
      Go to STEP 1
    end
end

3 The Tolerance Approach

In this section we consider the tolerance approach proposed by Wendell[4]. Given a linear program

$$\min \sum_{j=1}^{n} c_j x_j, \text{s.t.} \sum_{j=1}^{n} a_{ij} x_j = b_i, \qquad (9)$$
$$\text{for } i = 1, \ldots, m \text{ and } x_1, \ldots, x_n \geq 0,$$

we consider the following perturbed problem.

$$\min \sum_{j=1}^{n} (c_j + \alpha_j c_j^{'}) x_j, \ \text{s.t.} \sum_{j=1}^{n} a_{ij} x_j = b_i, \quad (10)$$
$$\text{for } i = 1, \ldots, m \text{ and } x_1, \ldots, x_n \geq 0$$

We denote $\mathbf{B}$ an optimal basis of (9) and IR the index set for the nonbasic variables. Let $\mathbf{y}_j = \mathbf{B}^{-1}\mathbf{a}_j$ as usual. Then $\mathbf{B}$ remains an optimal basis in (10) if

$$\sum_{i=1}^{m} (c_{B_i} + \alpha_{B_i} c_{B_i}^{'}) y_{ij} - (c_j + \alpha_j c_j^{'}) \leq 0 \qquad (11)$$

for each $j \in IR$, where $c_{B_i}, c_{B_i}^{'}, \alpha_{B_i}$ denote the corresponding basic variable coefficients in the vectors $c, c^{'}, \alpha$, repectively. Note that if $c^{'} = 1$(i.e., each $c_j^{'} = 1$), then α becomes an additive parameter and would represent additive variation in the coefficients. And if $c^{'} = c$, then α would represent multiplicative variation in the objective coefficient c. A maximum allowable tolerance on the *multiplicative* variations in the objective coefficients is denoted by α_*^{mult} and is given by

$$\alpha_*^{mult} = \text{minimum} \frac{c_j - z_j}{\sum_{i=1}^{m} | c_{B_i}^{'} y_{ij} | + | c_j^{'} |}$$
$$j \in IR^+$$

where $IR^+ = \{j \in IR : \sum_{i=1}^{m} | c_{B_i}^{'} y_{ij} | + | c_j^{'} | > 0\}$, see [1, 4] for more detail. A maximum allowable tolerance on the *additive* variations would then be

$$\alpha_*^{add} = \text{minimum} \frac{c_j - z_j}{\sum_{i=1}^{m} | y_{ij} | + | 1 |}$$
$$j \in IR^+$$

where $IR^+ = \{j \in IR : \sum_{i=1}^{m} | y_{ij} | + | 1 | > 0\}$.

For an illustration, consider the following problem given in [1].

$$\begin{array}{lll} \text{minimize} & -2x_1 + x_2 - x_3 & \\ \text{s.t.} & x_1 + x_2 + x_3 & \leq 6 \\ & -x_1 + 2x_2 & \leq 4 \\ & x_1, x_2, x_3 & \geq 0 \end{array} \qquad (12)$$

For this problem, the optimal simplex tableau is given by

	z	x_1	x_2	x_3	x_4	x_5	RHS
z	1	0	-3	-1	-2	0	-12
x_1	0	1	1	1	1	0	6
x_5	0	0	3	1	1	1	10

For the multiplicative variation, we get $\alpha_*^{mult} = \text{minimum}\{\frac{3}{3}, \frac{1}{3}, \frac{2}{2}\} = \frac{1}{3}$. This implies that the objective coefficients can vary within a tolerance of $33\frac{1}{3}$ percent and the given basis will remain optimal. The region is given by $T = [-\frac{8}{3}, -\frac{4}{3}] \times [\frac{2}{3}, \frac{4}{3}] \times [-\frac{4}{3}, -\frac{2}{3}] \times [0,0] \times [0,0]$ Note that when $c_4 = c_5 = 0$, no variation is permissible.

For an additive variation, we obtain $\alpha_*^{add} = \text{minimum}\{\frac{3}{5}, \frac{1}{3}, \frac{2}{3}\} = \frac{1}{3}$ and $S = [-\frac{7}{3}, -\frac{5}{3}] \times [\frac{2}{3}, \frac{4}{3}] \times [-\frac{4}{3}, -\frac{2}{3}] \times [-\frac{1}{3}, \frac{1}{3}] \times [-\frac{1}{3}, \frac{1}{3}]$

4 Expansion Technique on Additive Variation

In (11), we let $c^\circ = c$ and $c_{B_i}^{'} = c_j^{'} = 1$ for all i and j. Then, $\sum_{i=1}^{m}(c_{B_i}^\circ + \alpha) y_{ij} - (c_j^\circ + \alpha) \leq 0$, for each $j \in IR$. This gives

$$\sum_{i=1}^{m} y_{ij}\alpha - \alpha \leq c_j^\circ - \sum_{i=1}^{m} c_{B_i}^\circ y_{ij} = f(c^\circ) \qquad (13)$$

for each $j \in IR$. For the example in (12), the "critical" region of the parameter is

$$\begin{array}{cccccccc} \alpha_1 & -\alpha_2 & & & + & 3\alpha_5 & \leq & 3 \\ \alpha_1 & & -\alpha_3 & & + & \alpha_5 & \leq & 1 \\ \alpha_1 & & & -\alpha_4 & + & \alpha_5 & \leq & 2 \end{array}$$

where each column corresponds to the column of x_i.

Following the algorithm TAE in section 2.3, $\alpha_*^\circ = \frac{1}{3}$ and $z = (-\frac{5}{3}, \frac{4}{3}, -\frac{4}{3}, \frac{1}{3}, \frac{1}{3})$. Then, $c^1 = c^\circ + (c^\circ - z) = (-\frac{7}{3}, \frac{2}{3}, -\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3})$. This gives $S = [-\frac{7}{3}, -\frac{5}{3}] \times [\frac{2}{3}, \frac{4}{3}] \times [-\frac{4}{3}, -\frac{2}{3}] \times [-\frac{1}{3}, \frac{1}{3}] \times [-\frac{1}{3}, \frac{1}{3}]$.

To expand the region, we replace c° by c^1 in (13). Then the right-hand side vector becomes $(4, 2, \frac{7}{3})$. Again, $\alpha_*^1 = \frac{2}{3}$ and note that (II) is satisfied. Now, $c^2 = c^1 + (c^1 - z) = (-3, 0, 0, -1, -1)$ and trying to expand once more, the right-hand side vector gives $(6, 4, 3)$ and $\alpha_*^2 = 1$. At this point, note that $\alpha_*^2 < 2\alpha_*^1$. Hence, we stop the procedure of the expansion. The region R is obtained from (8), replacing y^i by c^2, as

$$R = [-3, -\frac{5}{3}] \times [0, \frac{4}{3}] \times [-\frac{4}{3}, 0] \times [-1, \frac{1}{3}] \times [-1, \frac{1}{3}]$$

Next, for the expansion of the right-hand side terms, we consider the following perturbed problem from (9):

$$\min \sum_{j=1}^{n} c_j x_j, \quad \text{s.t.} \quad \sum_{j=1}^{n} a_{ij} x_j = b_i + \beta_i b_i', \qquad (14)$$
$$\text{for } i = 1, \ldots, m \text{ and } x_1, \ldots, x_n \geq 0.$$

Let $\mathbf{B}$ be an optimal basis of (9), and the components of $\mathbf{B}^{-1}$ as B_{ij}^{-1} for $i, j = 1, \ldots, m$. Then, $\mathbf{B}$ will be an optimal basis for the above perturbed problem (14) if

$$\sum_{j=1}^{m} B_{ij}^{-1}(b_j + \beta_j b_j') \geq 0, \quad i = 1, \ldots, m \qquad (15)$$

This can be rewritten as

$$\sum_{j=1}^{m} (-B_{ij}^{-1} b_j') \beta_j \leq \bar{b}_j, \quad i = 1, \ldots, m \qquad (16)$$

where $\bar{\mathbf{b}} = \mathbf{B}^{-1}\mathbf{b}$. By letting $b_j' = 1$ for each j in (15) gives the additive variation, and the maximum allowable tolerances on the additive variation is given by

$$\beta_*^{add} = \text{minimum} \ \frac{\bar{b}_j}{\sum_{j=1}^{m} | -B_{ij}^{-1} |}$$
$$i \in IB^+,$$

where $IB^+ = \{j \in IB : \sum_{i=1}^{m} | -B_{ij}^{-1} | > 0\}$.

For the problem in (12), $b^\circ = b = (6, 4)$ and the optimal basis and its inverse are

$$\mathbf{B} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}, \quad \mathbf{B}^{-1} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

and the critical region of the parameter is

$$\begin{array}{cccc} -\beta_1 & & \leq & 6 \\ -\beta_1 & -\beta_2 & \leq & 10 \end{array}$$

Following the algorithm TAE in section 2.3, $\beta_*^\circ = 5$ and $z = (1, -1)$. Then, $b^1 = (11, 9)$. This gives $S = [1, 11] \times [-1, 9]$. To expand the region, we replace b° by b^1 in (16). This gives $\beta_*^1 = 10$ and note that (II) is satisfied. Now, $b^2 = (21, 19)$ and $R = [1, 21] \times [-1, 19]$. Replacing b° by b^2 in (16), we obtain $\beta_*^2 = 20$ which also satisfies (II). For this particular problem, this process can be continued and the final expanded region R can be given as

$$R = [1, +\infty) \times [-1, +\infty)$$

5 Concluding Remarks

While the tolerence approach is trying to find a range for which objective function coefficients and the right-hand side terms can be changed in a "one-time" basis, the expansion method, if possible, is trying to expand the region as much as possible and more than once from the result obtained by the tolerence approach.

Further work includes expansion technique on the multiplicative variations and implementation of this technique on a set of more realistic problems.

References

[1] Bazaraa, M.S., Jarvis, J.J. and Sherali, H.D., *Linear Programming and Network Flows*. 2nd ed., John Wiley & Sons, (1990).

[2] Dantzig, G.B., *Linear Programming and Extensions*, Princeton University Press, Princeton, N.J (1963).

[3] Wendell, R.E., "Using Bounds on the Data in Linear Programming: The Tolerance Approach to Sensitivity Analysis", *Mathematical Programming 29*, (1984), pp. 304-322.

[4] Wendell, R.E., "The Tolerance Approach to Sensitivity Analysis in Linear Programming", *Management Science*, Vol. 31, No. 5, (1985), pp. 564-578.