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Robust Stabilization of Time-delay Systems with Nonlinear Uncertainties

Juhyun Park and Sangchul Won

Dept. of Electrical Engr., POSTECH,
San 31 Hyoja Dong, Pohang, 790-784 Korea.
Tel:+82-562-279-2894, Fax:+82-562-279-2903, Email:juhyun@jane.postech.ac.kr

Abstract: This paper is concerned with the design of robust state feedback controller for a class of linear time-delay systems with norm-bounded nonlinear uncertainties. Under the proposed delay-independent criterion, asymptotic stability for the system is investigated using the conventional Lyapunov method. Moreover, the robust controller can be obtained by solving the linear matrix inequality which is equivalent to the suggestd conditions.

I. Introduction

The problem of the stability test and stabilization of time-delay systems has been explored over the decades. Many different approaches to solve the problem have been proposed [6,8,10,11,13]. However, accurate modeling of time-delay systems is not simple, so uncertain parameters and disturbances must be integrated into the model. Recently, the robust stabilization of a class of time-delay systems with linear time-varying uncertainties has been considered in [1,9,10]. In Cheres *et al.*[1], using Razumikhin stability theory[4] for robust controller design in conjunction with Lyapunov stability theory, a class of stabilizing nonlinear controllers for uncertain time-delay systems are synthesized and furthermore the control law developed is not continuous. In Shen *et al.* [10], the memoryless linear control law is presented to stabilize the uncertain system using Riccati equation approach. And Mahmoud[9] stabilized the system using two feedback controllers. One is linear memoryless and the other is of saturation type. In the case of control law of staturation type, the asymptotic stability of resulting closed-loop system is not guranteed.

In this paper, we investigate the robust stabilization of time-delay systems with nonlinear uncertainties using conventional Lyapunov function theory. In order to stabilize the system, a continuous memoryless state-feedback controller is presented. The control law consists of two parts. one is a linear memoryless control law which stabilizes the nominal sysetm and the other is a nonlinear continuous control law of saturation type which stabilizes the uncertain portion of the system. In the proposed condition, it is proved that the resulting closed-loop system with this control law is asymptotically stable in large. We may obtain the robust controller by solving the linear matrix inequality(LMI) which is equivalent to the proposed condition. In contrast to existing results on delay-independent robust stabilization, such as those in [1], [9] and [10], the LMI approach has the advantage that can be solved numerically very efficiently using interior-point methods[16]-[17].

The organization of this paper is as follows. We give the mathematical preliminaries in Section 2. In Section 3, the memoryless state-feedback controller to guarantee the exponential stability of the system is introduced. In Section 4, conclusions are provided.

Notation. The following notation will be used throughout the paper. R^n denotes the n dimensional Eulidean space, R^{nxm} is the set of all nxm real matrices, and $\|\cdot\|$ refers to either the Euclidean vector norm or the induced matrix 2-norm. Given an nxn matrix M, $\lambda_{min}[M]$ denotes the minimum eigenvalue of the matrix M. The notation $X > Y(X \geq Y)$, where X and Y are symmetric matrices, means that the matrix $X - Y$ is positive definite(positive semi-definite) matrix. A superscript T denotes the transpose of matrix(vector). I denotes n-dimensional identity matrix.

2. Robust stabilization

Consider a class of time-delay system with nonlinear uncertainties described by the following differential-difference equations:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + A_1 x(t-h) + F(t,x(t)) \\ &\quad + F_1(t,x(t-h)) + Bu(t), \\ x(t) &= \phi(t), \quad t \in [-h,0], \end{aligned} \tag{1}$$

where $x(t) \in R^n$ is the state vector, $u(t) \in R^m$ is the control input, the known constant matrices A, A_1, and B are of appropriate dimension, $h > 0$ is a delay time and $\phi(t)$ is a continuous vector-valued initial condition function. The plant uncertainties

F and F_1 are unknown and are used to represent nonlinear parameter perturbations with respect to the current state $x(t)$ and the delayed state $x(t-h)$ of the system, respectively. But F and F_1 are assumed to be bounded in magnitude. A system is said to be robustly stable if it is tolerant of change in certain specific bounds of perturbation. With the plant model given by (1), the problem is how to synthesize a continuous control law $u(t)$ that can guarantee asymptotic stability.

Now, we introduce the necessary assumption and lemma, with regard to the sysem given by (1), which will be used in the stability proof later.

Assumption 1: The matrix pair (A, B) is a uniformly completely controllable pair. And for all $(t, x) \in R\text{x}R^n$, there exist continuous matrix functions $f(\cdot)$ and $f_1(\cdot)$ which satisfy the following matching conditions[2]

$$F = Bf(t, x(t)), \quad F_1 = Bf_1(t, x(t-h)). \qquad (2)$$

Here, the nonlinear matrix funtions $f(t, x(t))$ and $f_1(t, x(t-h))$ are bounded, i.e.

$$\|f(t,x(t))\| \le \beta_0\|x(t)\|, \|f_1(t,x(t-h))\| \le \beta_1\|x(t-h)\| \qquad (3)$$

where β_0 and β_1 are given positive scalars.

It should be noted that the matching condition given in (2) is often not satisfied in many applications.

Lemma 1[5] *For given any constant $a > 0$ and any two suitably dimensioned matrices X, Y,*

$$XY^T + YX^T \le aXX^T + \frac{1}{a}YY^T. \qquad (4)$$

Under above assumptions, consider state-feedback controller to guarantee the asymtotic stability of time-delay systems with nonlinear uncertainties.

$$u(t) = u_1(t) + u_2(t), \qquad (5)$$

where

$$u_1(t) = -\frac{\gamma}{2}B^T Px(t), \qquad (6)$$

$$u_2(t) = -\frac{B^T Px\beta_0^2\|x\|^2}{\|B^T Px\|\beta_0\|x\| + \frac{\varepsilon}{2}\|x\|^2} \qquad (7)$$

where γ and ε are positive constants and P is a positive definite matrix.

Remark 1: A state-dependent function $\varepsilon\|x(t)\|^2$ is included into the control (7). The function is used to construct the continuous control law of saturation type in [15]. This is different either from [9] where a constant ε is used, or from [3] where a time function $\varepsilon e^{-\beta t}$ is employed.

To examine the stability of the system (1) with control input $u(t)$ (5), we define a Lyapunov function candidate $V(\cdot) : R^n\text{X}R_+ \to R_+$ as

$$V(x,t) = \frac{1}{2}\left(x^T(t)Px(t) + \int_{t-h}^{t} x^T(s)Rx(s)ds\right) \qquad (8)$$

where R is a positive definite matrix.

Then, based on consideration of the Lyapunov's direct method, we have the following theorem which shows asymptotic stability of dynamical sysem (1) with control input $u(t)$ given in (5).

Theorem 1: *Let us define M as*

$$M = \begin{bmatrix} -(A^TP + PA - (\gamma-1)PBB^TP + R) & -PA_1 \\ -A_1^TP & R - \beta_1^2 I \end{bmatrix} \qquad (9)$$

and suppose assumption 1 is valid. If there exist positive scslars ε, γ, and $(n\text{x}n)$ positive definite matrices P and R such that

$$\lambda_{min}(M) - \varepsilon > 0, \qquad (10)$$

i.e.,

$$\begin{aligned} M_1 &= R - \beta_1^2 I - \varepsilon I > 0 \qquad (11) \\ M_2 &= A^TP + PA - (\gamma-1)PBB^TP + R \\ &\quad + \varepsilon I + PA_1(R - \beta_1^2 I - \varepsilon I)^{-1}A_1^TP < 0 \qquad (12) \end{aligned}$$

then, the uncertain system (1) under the action of the state feedback controller given in (5) is asymptotically stable.

Proof: Let $x(t)$ be the solution of closed-loop dynamical system (1), and $V(t)$ denotes $V(x,t)$. The time derivative of $V(\cdot)$ along the trajectory of system (1) is given by

$$\begin{aligned} \dot V(t) &= \frac{1}{2}\left(\dot x^T(t)Px + x^TP\dot x + x^TRx - x_h^TRx_h\right) \\ &= -\frac{1}{2}X^T \begin{bmatrix} -(A^TP + PA + R) & -PA_1 \\ -A_1^TP & R \end{bmatrix} X \\ &\quad + x^TPBf + x^TPBf_1 + x^TPBu(t) \qquad (13) \end{aligned}$$

where $, x_h = x(t-h), X = [x^T(t)\ x_h^T]^T$.

Here, the follwing inequality holds

$$\begin{aligned} x^T(t)PBf_1 &\le \frac{1}{2}(x^T(t)PBB^TPx + f_1^Tf_1) \qquad (14) \\ &\le \frac{1}{2}(x^T(t)PBB^TPx + \beta_1^2x_h^Tx_h) \qquad (15) \end{aligned}$$

where Lemma 1 is used in the first inequality.

Then, using (6) and (15), we have the following

$$\dot V(t) \le -\frac{1}{2}X^TMX + \|B^TPx\|\beta_0\|x\| + x^TPBu_2. \qquad (16)$$

Substituting (7) and (9) into (16) and after some manipulation,

$$\dot{V}(t) \leq -\frac{1}{2}X^T MX + \frac{\|B^T Px\|\beta_0\|x\|\frac{\varepsilon}{2}\|x\|^2}{\|B^T Px\|\beta_0\|x\| + \frac{\varepsilon}{2}\|x\|^2}. \quad (17)$$

Here, the Hermitian block matrix M is positive definite if the conditions (11) and (12) is hold[7]. Then, using Rayleigh's coefficient [14] and using the inequality $ab/(a+b) \leq a,\ \forall\ a,b > 0$, we obtain a new upper bound on $\dot{V}(t)$,

$$\begin{aligned}\dot{V}(t) &\leq -\frac{1}{2}\lambda_{min}(M)\|X\|^2 + \frac{\varepsilon}{2}\|x\|^2 \\ &\leq -\frac{1}{2}\lambda_{min}(M)\|x\|^2 + \frac{\varepsilon}{2}\|x\|^2. \quad (18)\end{aligned}$$

Here, the condition (10) is satisfied, we have $\dot{V}(t) < 0$ and we conclude that the closed-loop trajectories of the uncertain system (1) under the action of the control law (5) are asymptotically stable.

In above Theorem, if we choose the matrix R of Lyapunov function given in (8) as $R = (1+\beta_1^2+\varepsilon)I$, we have the following corollary.

Corollary 1: *Assume that there exist a positive definite matrix P and a positive number γ which satisfy the following inequality:*

$$A^T P + PA - P((\gamma-1)BB^T - A_1A_1^T)P + (1+\beta_1^2+2\varepsilon)I < 0 \quad (19)$$

Then with the state feedback controller $u(t)$ given in (5), the uncertain system (1) is asymptotically stable.

Proof: The proof is obvious, and is omitted.

Remark 2. The control (5) consists of two parts $u_1(t)$ and $u_2(t)$. Here, u_1 is a linear state feedback controller which stabilizes the nominal system and the delayed perturbation $F_1(\cdot)$, and u_2 is continuous nonlinear state feedback controller which is used to compensate for the system uncertainty $F(\cdot)$ in system (1) to produce an asymptotic stability.

Remark 3. The criteria for robust stabilization given in (11) and (12) are mainly affected by A_1 and β_1. Upper bound β_0 does not affect the criteria directly but can cause gain of u_2 to become large.

In Corollary 1, we can convert the condition (19) to the following LMI problem which is equvalent to the condition. For details of LMI optimization problem, see [16], [17], and refences therein.

Corollary 2. *The inequality (19) is equivalent to the LMI*

$$\Omega(X,\varepsilon,\gamma) < 0 \quad (20)$$

where

$$\Omega(X,\varepsilon,\gamma) = \begin{bmatrix} XA^T + AX + G & XH^{1/2} & \sqrt{2}X & B \\ H^{1/2}X & -I & 0 & 0 \\ \sqrt{2}X & 0 & -\varepsilon^{-1}I & 0 \\ B^T & 0 & 0 & \gamma^{-1}I \end{bmatrix}$$

$$G = BB^T + A_1A_1^T,\quad H = (1+\beta_1^2)I,\quad X = P^{-1}$$

Proof: The above LMI is equvalent to the following matrix inequality [7]

$$\begin{aligned} XA^T &+ AX - ((\gamma-1)BB^T - A_1A_1^T) \\ &+ (1+\beta_1^2)XX + 2\varepsilon XX < 0. \end{aligned}$$

If we prmultiply and postmultiply by P^{-1}, then there results (19).

Remark 4. Corollay 2 provides a delay-independent condition for robust stabilization of linear uncertain time-delay systems in terms of the solvability of a linear matric inequality. This is in contrast with the result of [1], [9] and [10] which developed delay-independent criteria for robust stability of time-delay systems in terms of the solution of either a Lyapunov or Riccati equation. The robust criterion of Corollay 2 can be tested numerically very efficiently using interior point algorithms[16-17], which have been recently developed for solving linear matrix inequalities.

3. Example

In this section, we illustrate an example to show that the proposed controller gurantee the robust stability.

Consider a time-delay dynamical system with uncertainties described by

$$\dot{x}(t) = Ax(t) + A_1(t-h) + F(t,x(t)) + F_1(t,x_h) + Bu(t)$$

where

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\quad A_1 = \begin{bmatrix} 0.5 & 0.3 \\ 0.3 & 0.5 \end{bmatrix},\quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

with $x(t) = [1 \quad -0.5]^T$ for $-1 \leq t < 0$, and the delay h is 1. In simulation, we assume the following representation of uncertain functions:

$$f(t,x) = 0.5\sin x_2^2,\quad f_1(t,x_h) = 0.5\sin(x_1(t-h)x_2(t-h)).$$

f and f_1 are assumed to be bounded by $\beta_0 = 0.5$ and $\beta_1 = 0.5$, respectively.

The design goal of this example is to stabilize the system by the state feedback control law (5).

The solutions of the LMI (20) are given by

$$P = \begin{bmatrix} 4.535 & 1.2156 \\ 1.2156 & 0.7188 \end{bmatrix},$$

$$\gamma = 21.5523,\quad \varepsilon = 0.0635.$$

Then, the control law (5) is

$$u(t) = -(13.1x_1 + 7.7457x_2) - \frac{0.5\sqrt{1.4777x_1^2 + 0.5167x_2^2}\sqrt{x} + 0.03175\sqrt{x}}{(0.3039x_1 + 0.1797x_2)\sqrt{x}}$$

where $\sqrt{x} = \sqrt{x_1^2 + x_2^2}$.
By this, it can be conclude that the system of the example is exponentially stable by the control laws (6) and (7). To comfirm this, the results of the simulation of this example are depicted in Figures 1 and 2. It is shown from Figure 1 that the closed-loop system is indeed exponentially stable. And it is obvious from Figure 2 that the control proposed in this paper is continuous, and no chattering will appear for the control.

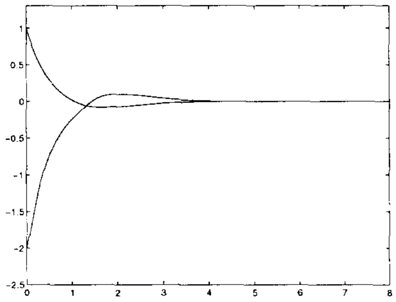

Figure 1: Closed-loop state trajectories

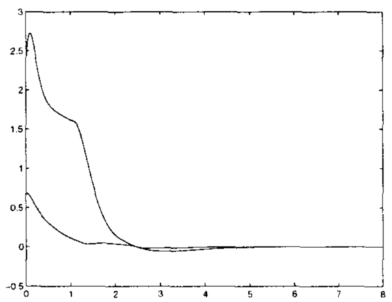

Figure 2: Control trajectories

4. Conclusions

The robust control problem of a class of time-delay systems in the presence of nonlinear uncertainties has been discussed. Based on the Lyapunov method, we propose an approach to synthesizing a class of continuous state feedback controller. The conditions under which the state trajectory is asymptotically stable are proposed. Moreover, it is shown that the proposed conditions are equivalent to the LMI problems. The problem can be easily solved by numerical method developed recently.

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