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Discrete-Time BLUFIR Filter

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Abstracts A new version of the discrete-time optimal FIR (finite impulse response) filter utilizing only the measurements of finite sliding estimation window is suggested for linear time-invariant state-space models. This filter is called the BLUFIR (best linear unbiased finite impulse response) filter since it provides the BLUE (best linear unbiased estimate) of the state obtained from the measurements of the estimation window. It is shown that the BLUFIR filter has the deadbeat property when there are no noises in the estimation window.

Keywords Filter, BLUE, FIR, Deadbeat property

1. Introduction

Recently, Kwon *et al.* have suggested the optimal FIR filter [1, 2] for state-space models with system and measurement noises as a generalization of the limited memory filters [3, 4, 8]. The limited memory filters are known to be robust to modeling errors since they utilize finite measurements over the most recent time interval (estimation window) [9]. In addition, it is known in general that the FIR filter structure is robust to numerical errors such as the roundoff errors and the coefficient quantization errors compared to the IIR (infinite impulse response) or recursive filter structure [10]. Thus, the optimal FIR filter has been used as an alternative to the Kalman filter [5] when the performance of the latter degrades due to modeling errors or numerical errors [6].

In order to determine the impulse response of the optimal FIR filter, *a priori* information about initial state of each estimation window is required as well as the state-space model and the covariances of the system and measurement noises. It is, however, impractical to assume that the state information is given at each time. Moreover, a priori information is hardly obtained in some situations such that the system is in an abrupt change or not asymptotically stable. Hence, in that case, the optimal FIR filter of [1, 2] has been derived by taking each initial state covariance as infinity, and it is shown to have some good properties. It is always time-invariant and needs no *a priori* information. It has the deadbeat property for noise-free systems. However, it has some disadvantages. Firstly, its estimation error covariance is relatively larger than other filters using initial information. Secondly, it cannot be applied to singular systems.

In this paper, we propose a new version of the optimal FIR filter that does not require *a priori* information either, but has smaller error covariance than the existing optimal FIR filter using no initial information, while other good properties are maintained. The proposed filter provides the BLUE (best linear unbiased estimate) of the state by combining the BLUE of the initial state of the estimation window and the optimal FIR filter for the residual system generated by the BLUE of the initial state. This FIR filter will be called the BLUFIR (best linear unbiased finite impulse response) filter hereafter, in the sense that it has the smallest error covariance among the FIR filters utilizing only the measurements of finite interval and its estimate is unbiased.

Since the initial state is estimated from the measurements at each window, the BLUFIR filter does not require *a priori* information. Also, it is time-invariant and has the deadbeat property similarly as the existing optimal FIR filter using no initial information. Moreover, it can be directly applied to the singular systems. The only disadvantage is that it requires more computations to determine the impulse response than the existing optimal FIR filter. However, in many cases, it does not matter because the filter is time-invariant and the impulse response can be determined in advance by off-line computations.

This paper is organized as follows. In Section 2, the BLUFIR filter is proposed and its properties are discussed. Finally, the conclusion will be made in Section 3.

2. BLUFIR Filter

Let us consider the linear time-invariant discrete-time state-space model:

$$\begin{aligned} x(t+1) &= Ax(t)+Gw(t), \\ y(t) &= Cx(t)+v(t) \end{aligned} \tag{1}$$

,where $x(\cdot)$ ($\in \Re^n$) is the state and $y(\cdot)$ ($\in \Re^q$) is the observation. The system noise $w(\cdot)$ ($\in \Re^p$) and the measurement noise $v(\cdot)$ ($\in \Re^q$) are zero-mean white Gaussian and mutually uncorrelated. The covariances of $w(\cdot)$ and $v(\cdot)$ are Q and R, respectively.

In the estimation window $[i-M, i]$, the system (1) can be represented into the vector regression form:

$$Y_M(i) = L_M x(i-M) + G_M W_M(i) + V_M(i) \quad (2)$$

,where:

$$\begin{aligned} Y_M(i) &= [y(i-M)^T \ y(i-M+1)^T \ \cdots \ y(i)^T]^T, \\ W_M(i) &= [w(i-M)^T \ w(i-M+1)^T \ \cdots \ w(i)^T]^T, \\ V_M(i) &= [v(i-M)^T \ v(i-M+1)^T \ \cdots \ v(i)^T]^T \end{aligned}$$

and

$$L_M = \begin{pmatrix} C \\ CA \\ CA^2 \\ \cdot \\ \cdot \\ CA^M \end{pmatrix}, \quad dim(L_M) = (M+1)q \times n, \quad (3)$$

$$G_M = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 \\ CG & 0 & 0 & 0 & \cdots & 0 \\ CAG & CG & 0 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & 0 \\ CA^{M-1}G & CA^{M-2}G & \cdot & \cdot & CG & 0 \end{pmatrix}, \quad (4)$$

$$dim(G_M) = (M+1)q \times (M+1)p.$$

When *a priori* information on the initial state $x(i-M)$ of the estimation window is available, i.e. a mean of initial state $\bar{x}(i-M)$ and its covariance $P(i-M, i-M)$ are known, *a priori* residual of the estimation window $O_M(i)$ can be defined as:

$$\begin{aligned} O_M(i) &= [o(i-M)^T \ o(i-M+1)^T \ \cdots \ o(i)^T]^T \\ &= Y_M(i) - L_M \bar{x}(i-M). \quad (5) \end{aligned}$$

Then, *a priori* residual of the estimation window can be represented by the output of the following zero-mean *a priori* residual system defined by:

$$\begin{aligned} x_o(j+1) &= Ax_o(j) + Gw(j), \quad (i-M \le j \le M) \\ o(j) &= Cx_o(i) + v(j) \quad (6) \\ &,\text{where} \quad x_o(i-M) = x(i-M) - \bar{x}(i-M). \end{aligned}$$

The initial state covariance of the *a priori* residual system is $P(i-M, i-M)$.

Based upon the measurements and a priori information on the initial state of the estimation window, the optimal estimate of the state $x(i)$ is given by:

$$\hat{x}_{[i;M]}(i) = A^M \bar{x}(i-M) + H_{[i;M]} O_M(i) \quad (7)$$

,where $H_{[i;M]}$ is the optimal FIR filter [1] given in the form:

$$H_{[i;M]} = [\ h_{[i;M]}(M) \ \ h_{[i;M]}(M-1) \ \ \cdots \ \ h_{[i;M]}(0)\ ].$$

The impulse response of the optimal FIR filter can be determined when $\{A, C\}$ is observable and $M \ge n-1$, by the fast algorithm [2]:

$$h_{[i;M]}(j) = \Phi_i(j)\Omega_i(M-j)C^T R^{-1}, \quad (0 \le j \le M) \quad (8)$$

,where $\Phi_i(\cdot)$ and $\Omega_i(\cdot)$ are obtained by:

$$\Phi_i(n+1) = \Phi_i(n)[I - \Omega_i(M-n)C^T R^{-1} C]A, \quad (9)$$
$$(0 \le n \le M),$$

$$\Phi_i(0) = I, \quad (10)$$

$$\Omega_i(n) = \Omega_i^-(n) - \Omega_i^-(n)C^T[C\Omega_i^-(n)C^T + R]^{-1} \cdot C\Omega_i^-(n), \quad (11)$$

$$\Omega_i^-(n+1) = A\Omega_i(n)A^T + GQG^T, \quad (0 \le n \le M) \quad (12)$$

$$\Omega_i^-(0) = P(i-M, i-M). \quad (13)$$

The estimation error covariance of the optimal FIR filter is given by the M-step solution $\Omega_i(M)$ of the discrete Ricatti equation(11) [1].

However, *a priori* information may be hardly obtained in some situations such that the system is in an abrupt transition or not asymptotically stable. Moreover, it is impractical to assume that *a priori* information can be given at each time. Thus, the optimal FIR filter using no initial information [1] was suggested. This optimal FIR filter is obtained by assuming the initial state covariance $P(i-M, i-M)$ as ∞I and the initial state mean $\bar{x}(i-M)$ as zero. Though it is a suboptimal filter, it has many advantages in practical applications. It needs no *a priori* information. It is time-invariant and thus, its impulse response can be obtained in off-line computations. It has the deadbeat property i.e. it gives exact state when there are no noises in the estimation window [?].

However, it has two disadvantages. Firstly, its error covariance is relatively large compared to the optimal FIR filters using *a priori* information. The error covariance of the optimal FIR filter is given by the finite step solution of the discrete Ricatti equation (11),(12). Thus, the monotonicity of the Ricatti equation [12] about the initial value implies that the optimal FIR filter using no initial information has always greater error covariance than other optimal FIR filters. Secondly, its impulse response cannot be obtained when the matrix A is singular.

In this paper, we propose a new version of the optimal FIR filter that does not require *a priori* information either, but has smaller error covariance than the existing optimal FIR filter using no initial information, while other good properties are maintained. Unlike the optimal FIR filter using no initial information, it utilizes the BLUE (best linear unbiased estimate) scheme [11] to provide optimally noise-suppressed and unbiased estimate of the initial state at each estimation window. The proposed filter provides the BLUE of the current state by combining the BLUE of the initial state of the estimation window and the optimal FIR filter for the residual system generated by the BLUE of the initial state. This FIR filter will be called the BLUFIR (best linear unbiased finite impulse response) filter hereafter in the sense that it has the smallest error covariance among the FIR filters utilizing only the measurements of finite interval and its estimate is unbiased.

The BLUFIR filter is defined by the filter equation:

$$\hat{x}_B(i) = H_B Y_M(i) = \sum_{k=0}^{M} h_B(k) y(i-k), \qquad (14)$$

,where:

$$H_B = [\ h_B(M)\ \ h_B(M-1)\ \ \cdots\ \ h_B(0)\ ]$$

and the performance criterion:

$$E[\ (x(i) - \hat{x}_B(i))^T (x(i) - \hat{x}_B(i))\ |\ x(i-M) = ?\ ]. \quad (15)$$

To provide an optimally noise-suppressed and unbiased estimate of the initial information, the BLUE scheme [11] is adopted. Now, consider the linear regressed model of the system (2). Since, $W_M(i)$ and $V_M(i)$ are merely noises, the BLUE estimate of the initial state at the estimation window $\bar{x}^B(i-M)$ can be given, when $\{A, C\}$ is observable, by the following equation:

$$\begin{aligned} \bar{x}_B(i-M) &= Z_M Y_M(i), \\ Z_M &= (L_M^T \Xi_M^{-1} L_M)^{-1} L_M^T \Xi_M^{-1} \end{aligned} \qquad (16)$$

,where Ξ_M is the covariance of the noise $G_M W_M(i) + V_M(i)$ given by:

$$\Xi_M = G_M \begin{pmatrix} Q & 0 & \cdots & 0 \\ 0 & Q & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & \cdots & 0 & Q \end{pmatrix} G_M^T + \begin{pmatrix} R & 0 & \cdots & 0 \\ 0 & R & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & \cdots & 0 & R \end{pmatrix}. \qquad (17)$$

The error covariance of $\bar{x}_B(i-M)$ is given by [11]:

$$P_B(i-M, i-M) = (L_M^T \Xi_M^{-1} L_M)^{-1}. \qquad (18)$$

When $\bar{x}(i-M)$ is treated as a mean of the initial state, the error covariance $P_B(i-M, i-M)$ becomes the covariance of the initial state. Then, an optimal FIR filter can be derived via the equation (8) by using $P_B(i-M, i-M)$ as the initial state covariance $P(i-M, i-M)$. Let us denote this optimal FIR filter by $\tilde{H}_B$. This is the optimal filter for the zero-mean residual system generated from the $\bar{x}_B(i-M)$. Then, from (7) and (16), the BLUFIR filter, H_B is obtained from $\tilde{H}_B$ and Z_M as:

$$H_B = \tilde{H}_B + (A^M - \tilde{H}_B L_M) Z_M. \qquad (19)$$

The BLUFIR filter doesn't require *a priori* information about the state. Further, it is always time-invariant even for nonstationary systems and it can be obtained even for the singular systems.

In the following theorem, it will be shown that the BLUFIR filter is the best FIR filter when there is no *a priori* information.

Theorem 2.1 Among the FIR filters not utilizing *a priori* information on the state, the BLUFIR filter provides the smallest error covariance.

proof: The error covariance of the optimal FIR filter is the solution of the finite step discrete Ricatti equation (11),(12). Let $\Omega_i^1(M)$ and $\Omega_i^2(M)$ are the two solutions of the discrete Ricatti equations whose initial values are $P_1(i-M, i-M)$ and $P_2(i-M, i-M)$, respectively. The monotonicity of the discrete Ricatti equation [12] assures the following inequality:

$$\Omega_i^1(M) \geq \Omega_i^2(M)$$
$$\text{if } P_1(i-M, i-M) \geq P_2(i-M, i-M).$$

Thus, the BLUFIR filter provides the smallest error covariance, since the BLUE error covariance $P_B(i-M, i-M)$ is the smallest covariance of the initial state that can be obtained from the observation $Y_M(i)$. This completes the proof. ■

In the following theorem, it will be shown that the BLUFIR filter has the deadbeat property and is an unbiased estimator.

Theorem 2.2 The BLUFIR estimate $\hat{x}_B(i)$ is exact when there are no noises in the estimation window $[i-M, i]$ and is unbiased when there are noises.

proof: When there are no noises in the estimation window $[i-M, i]$, the observation $Y_M(i)$ and the state $x(i)$ is determined by the initial state $x(i-M)$ as:

$$\begin{aligned} Y_M(i)|\{\text{no noises}\} &= L_M x(i-M), \\ x(i)|\{\text{no noises}\} &= A^M x(i-M). \end{aligned}$$

Then, $\bar{x}_B(i-M)$ becomes $x(i-M)$ since:

$$\begin{aligned} &\bar{x}_B(i-M)|\{\text{nonoises}\} = Z_M Y_M(i)|\{\text{no noises}\} \\ &\quad = (L_M^T \Xi_M^{-1} L_M)^{-1} L_M^T \Xi_M^{-1} L_M x(i-M) \\ &\quad = x(i-M). \end{aligned}$$

In the sequel, $\hat{x}_B(i)$ becomes $x(i)$ since:

$$\begin{aligned} &\hat{x}_B(i)|\{\text{no noises}\} = H_B Y_M(i)|\{\text{no noises}\} \\ &= [\tilde{H}_B + (A^M - \tilde{H}_B L_M) Z_M] Y_M(i)|\{\text{no noises}\} \\ &= \tilde{H}_B [Y_M(i)|\{\text{no noises}\} \\ &\quad - L_M Z_M Y_M(i)|\{\text{no noises}\}] \\ &\quad + A^M Z_M Y_M(i)|\{\text{no noises}\} \\ &= A^M x(i-M) \\ &= x(i)|\{\text{no noises}\}. \end{aligned} \qquad (20)$$

From (20), the following identities hold:

$$H_B L_M = A^M. \qquad (21)$$

When there are noises, $\hat{x}_B(i)$ is unbiased, since:

$$E[\hat{x}_B(i) - x(i)] = H_B L_M x(i-M) - A^M x(i-M) = 0.$$

This completes the proof. ■

In the theorem 2.1 and 2.2, it is shown that the BLUFIR filter is the best linear unbiased filter utilizing only the measurements of the estimation window.

The only disadvantage of the BLUFIR filter compared to the optimal FIR filter using no initial information is that it requires more computations to determine the filter impulse response. To determine the BLUFIR filter, the computations of the order q^3M^3 for the BLUE of the initial information are added to the computations of the order Mn^3 for the optimal FIR filter. However, in many cases, it will not matter because the filter impulse response can be determined in off-line computations and the on-line filter equation (14) is the same as that of other optimal FIR filters.

To compare the estimation error covariances of the BLUFIR filter and the optimal FIR filter using no initial information, they are applied to the state model (1) whose parameters are:

$$A = \begin{pmatrix} \cos(\pi/16) & \sin(\pi/16) \\ -\sin(\pi/16) & \cos(\pi/16) \end{pmatrix}, \; C = [1\ 0], \; G = [1\ 1]^T.$$

The system noise covariance Q is 0.1 and the measurement noise covariance R is 0.1. In this example, as a norm, the maximum singular value of the error covariance matrix is used. In Figure 1, the norms of the error covariance matrices of the filters are plotted. In this figure, we can see that the BLUFIR filter always has smaller error covariance compared to the optimal FIR filter using no initial information. The difference is dominant when the filter length is short. As the filter length grows, the error covariances converges to the solution of the DARE (discrete algebraic Ricatti equation) of the system, that is the error covariance of the steady-state Kalman filter.

3. Conclusion

The BLUFIR filter is proposed as a new version of the optimal FIR filter. It gives the BLUE (best linear unbiased estimate) of the state and doesn't require *a priori* information. It provides the following properties. It is time-invariant and has the deadbeat property. It can be obtained even for the singular systems unlike the existing optimal FIR filter using no initial information. It is robust to modeling errors and numerical errors. Therefore, it is expected that the suggest BLUFIR filter will be used in many fields, especially in the areas that the performance of the Kalman filter degrades.

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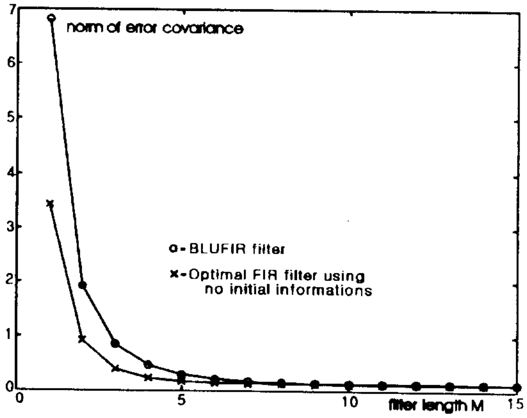

Figure 1: comparison of the error covariances