Proceedings of the 11th
KACC, October 1996

A Theory of Linear Quasi-Time Invariant Filters

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Abstracts In this paper, the eigenstructure of a class of linear time varying systems, termed as linear quasi-time invariant(LQTI) systems, is investigated. A system composed of dynamic devices such as linear time varying capacitors and resistors can be an example of the class. To effectively describe and analyze the LQTI systems, a generalized differential operator G is introduced. Then the dynamic systems described by the operator G are studied in terms of eigenvalue, frequency characteristics, stability and an extended convolution. Some basic attributes of the operator G are compared with those of the differential operator D. Also the corresponding generalized Laplace transform pair is defined and relevant properties are derived for frequency domain analysis of the systems under consideration. As an application example, a LQTI circuit is examined by using the concept of eigenstructure of LQTI system. The LQTI filter processes the sinusoidal signals modulated by some functions.

Keywords eigenvalue problems, frequency analysis, linear quasi-time invariant systems, exponential order

1. INTRODUCTION

In the theory of linear time invariant(LTI) systems, the eigenstructure is a very important concept for analysis and applications. By virtue of the eigenstructure, the linear time invariant systems can be analyzed by simple algebraic methods and their results are easily applicable to various kinds of engineering areas such as system control, circuit and filter design, signal processing and communication. The eigenstructure of LTI systems is derived from the characteristics of the differential operator D which is known to be the most basic operator to describe dynamical phenomena of the nature. The eigenstructure of the LTI systems are described by eigenvalues and eigenvectors.

In this paper, we define a new operator G composed of the differential operator D and some given time functions to describe dynamical phenomena of the nature. It will be shown that LQTI systems are described by constant system matrix, that is, the LQTI systems described by G operators are shown to have the same eigenstructure as the case of LTI systems described by D. It will be shown that LQTI systems can be analyzed in a similar way as in LTI systems by using the relation of eigenvalue and eigenvector in the time domain. A point of the complex s-plane in the traditional Laplace transformation represents an complex exponential function, while a point in the extended Laplace transform space has a different meaning. In case of the latter space, a point represents an complex exponential function modulated by some function.

The convolution is a basic operation for LTI systems, representing a filter. In this paper, an extended convolution for the LQTI systems is also developed. And the LQTI circuit device is defined with the purpose to construct the LQTI filter. Finally, we give an example of a physical LQTI filter. The LQTI filter is very different from the LTI filter and has many useful properties in comparison with a Wiener filter or a Kalman filter.

2. LINEAR QUASI TIME-INVARIANT SYSTEMS

In this section, a dynamic operator G is introduced to handle a class of linear time varying systems called LQTI systems. It will be shown that the LQTI systems have invariant eigenvalues regardless of coordination transform, and that analytic solutions of the LQTI systems can be obtained. To be specific, let the operator G be defined as

$$\begin{aligned} Gx(t) &= g_1(t)\frac{d}{dt}x(t) + g_0(t)x(t) \\ &= (g_1(t)D + g_0(t))x(t) \end{aligned} \tag{1}$$

where $g_1(t)$ and $g_0(t)$ are arbitrarily given functions with $g_1(t) =! 0$ and $x(t)$ is a variable. In the sequel, the independent variable t may be dropped for simplicity when no confusions arise. Obviously, the operator G is linear. The operator G has the following properties corresponding to those of the differential operator D.

Theorem 2.1 : Suppose $Gx = \lambda x$ for some x and some constant λ. Then, for any non-negative integer n,

$$G^n x = \lambda^n x ,$$

where $G^i x = G(G^{i-1}x), i = 2,3,\cdots,n$, and $G^1 x = Gx$.
Note that the solution of the differential equation $Gx(t) = \lambda x(t)$ is $x(t) = c\exp(\int_0^t (\lambda - g_0)/g_1\, d\tau)$, where c is a constant determined by the given initial condition. Since the operator G is linear, we can easily prove the above property by the induction method.

Theorem 2.2 : For $a < b$ let $G_{[a,b]}^{-1}$ denote the inverse of the operator G such that

$$G_{[a,b]}^{-1} y(t) = e^{-\int_0^t \frac{g_0(\tau)}{g_1(\tau)}d\tau} \left(\int_a^b \frac{y(\tau)}{g_1(\tau)} e^{\int_0^\tau \frac{g_0(\xi)}{g_1(\xi)}d\xi} d\tau\right) \tag{2}$$

Then, for some function $x(t)$, $GG_{[0,t]}^{-1}x = x$, for all $t > 0$, if $G_{[0,t]}^{-1}x$ exists. Also, if $x(0) = 0$, then $G_{[0,t]}^{-1}Gx = x$ for all $t > 0$, i.e., the operator G and $G_{[0,t]}^{-1}$ commute if $x(0) = 0$.

Proof : The first property can be proved by showing that $GG_{[0,t]}^{-1}x = x$. From (2)

$$\begin{aligned} G(G_{[0,t]}^{-1}x) &= G(e^{-\int_0^t \frac{g_0}{g_1}d\tau} \int_0^t \frac{x}{g_1} e^{\int_0^\tau \frac{g_0}{g_1}d\xi} d\tau) \\ &= g_1(-\frac{g_0}{g_1} e^{-\int_0^t \frac{g_0}{g_1}d\tau} \int_0^t \frac{x}{g_1} e^{\int_0^\tau \frac{g_0}{g_1}d\xi} d\tau + e^{-\int_0^t \frac{g_0}{g_1}d\tau} \frac{x}{g_1} e^{\int_0^t \frac{g_0}{g_1}d\tau}) \\ &\quad + g_0(e^{-\int_0^t \frac{g_0}{g_1}d\tau} \int_0^t \frac{x}{g_1} e^{\int_0^\tau \frac{g_0}{g_1}d\xi} d\tau) \\ &= x(t) \end{aligned}$$

Therefore, $G_{[0,t]}^{-1}$ is the inverse of G for all $t > 0$. Next, to prove the second property that $G_{[0,t]}^{-1}Gx = x$ with $x(0) = 0$, observe that Eq. (2) implies

$$
\begin{aligned}
G_{[0,t]}^{-1}(Gx) &= e^{-\int_0^t \frac{g_0}{g_1} d\tau}\left(\int_0^t \frac{Gx}{g_1} e^{\int_0^\tau \frac{g_0}{g_1} d\xi} d\tau\right) \\
&= e^{-\int_0^t \frac{g_0}{g_1} d\tau}\left(\int_0^t \dot{x} e^{\int_0^\tau \frac{g_0}{g_1} d\xi} d\tau + \int_0^t \frac{g_0}{g_1} x e^{\int_0^\tau \frac{g_0}{g_1} d\xi} d\tau\right) \\
&= x(t) - x(0) e^{-\int_0^t \frac{g_0}{g_1} d\tau} \\
&= x(t) \qquad \text{if } x(0) = 0
\end{aligned}
$$

which means that G and $G_{[0,t]}^{-1}$ commute for all $t > 0$ if $x(0) = 0$. QED.

Consider now the following homogeneous differential equation expressed in terms of the operator G:

$$G^n x + a_{n-1} G^{n-1} x + \cdots + a_1 G x + a_0 x = 0, \tag{3}$$

where $a_i, (i = 0,1,2,\cdots,n-1)$ are constant coefficients. From the linearity and Theorem 2.1, we find that $Gx = \lambda x$ implies

$$\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0 = 0 \tag{4}$$

The characteristic equation of the differential equation (3) has apparently the same form as that of the LTI differential equations. However, the physical meaning can be quite different from each other.

To obtain a closed form solutions of Eq. (3) to handle LQTI systems, define the state vectors as follows:

$$
\begin{aligned}
x_1 &= x, \\
x_2 &= Gx_1, \\
x_3 &= Gx_2, \\
&\ \vdots \\
x_n &= G^{n-1} x
\end{aligned}
\tag{5}
$$

Then

$$Gx_n = -a_{n-1}x_n - \cdots - a_0 x_1. \tag{6}$$

Hence, we find from Eq. (3) that

$$G\mathrm{x} = A\mathrm{x}, \qquad \mathrm{x}(t_0) = \mathrm{x}_0, \tag{7}$$

where

$$
\begin{aligned}
\mathrm{x} &= [x_1, \cdots, x_n]^t, \\
G\mathrm{x} &= g_1(t) D\mathrm{x} + g_0(t)\mathrm{x}, \\
D\mathrm{x} &= [Dx_1, \cdots, Dx_n]^t,
\end{aligned}
$$

$$
A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix}
$$

In the above, the superscript t denotes the transpose.

Theorem 2.3 : Suppose that the eigenvalues of the matrix A in Eq. (7) are distinct. Then, the solution of Eq. (7) is given as follows :

$$\mathrm{x}(t) = PQ(t)P^{-1}\mathrm{x}(t_0) \tag{8}$$

where

$$Q(t) = diag\left(e^{\int_0^t \frac{\lambda_1 - g_0(\tau)}{g_1(\tau)} d\tau}, e^{\int_0^t \frac{\lambda_2 - g_0(\tau)}{g_1(\tau)} d\tau}, \cdots, e^{\int_0^t \frac{\lambda_n - g_0(\tau)}{g_1(\tau)} d\tau}\right) \tag{9}$$

$$P^{-1}AP = diag(\lambda_1, \lambda_2, \cdots, \lambda_n)$$

In case when A involves multiple eigenvalues λ_i with multiplicity m_i, where $i = 1,2,\cdots,k$, and $m_1 + m_2 + \cdots + m_k = n$, and the eigenvectors corresponding to λ_i are also multiple with the corresponding multiplicity m_i, then

$$\mathrm{x}(t) = PS(t)Q(t)P^{-1}\mathrm{x}(t_0) \tag{10}$$

where

$$Q(t) = diag(Q_1(t), Q_2(t), \cdots, Q_k(t)) \tag{11}$$

with

$$
\begin{aligned}
Q_i(t) &= diag\left(e^{\int_0^t \frac{\lambda_i - g_0(\tau)}{g_1(\tau)} d\tau}, e^{\int_0^t \frac{\lambda_i - g_0(\tau)}{g_1(\tau)} d\tau}, \cdots, e^{\int_0^t \frac{\lambda_i - g_0(\tau)}{g_1(\tau)} d\tau}\right) \\
&= m_i \times m_i \text{ matrix}
\end{aligned}
\tag{12}
$$

and

$$S(t) = diag(S_1(t), S_2(t), \cdots, S_k(t)) \tag{13}$$

with

$$
S_i(t) = \begin{bmatrix} 1 & \alpha(\mathbf{t}) & \frac{\alpha(\mathbf{t})^2}{2!} & \cdots & \frac{\alpha(\mathbf{t})^{m_i - 1}}{(m_i - 1)!} \\ 0 & 1 & \alpha(\mathbf{t}) & \cdots & \frac{\alpha(\mathbf{t})^{m_i - 2}}{(m_i - 2)!} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \alpha(\mathbf{t}) \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} = m_i \times m_i \text{ matrix} \tag{14}
$$

where

$$\alpha(t) = \int_0^t \frac{1}{g_1(\tau)} d\tau \tag{15}$$

The proof of the above Theorem 2.3 is given in Appendix A.

3. EXTENDED LAPLACE TRANSFORM

The stability of the LQTI systems is determined by the roots of characteristic equation or the eigenvalues of the system matrix. The LQTI systems can also be analyzed in the frequency domain as in LTI systems. For this purpose, an extended Laplace transform is introduced. This transform enables us to convert the LQTI systems with initial conditions into simple algebraic equations.

Definition 3.1 : Extended Laplace transform pair.
Let $x(t)$ be a function on $[0, \infty)$. For arbitrarily given functions $g_0(t)$ and $g_1(t)$ with $1/g_1(t) > 0$, the extended Laplace transform of $x(t)$ and its inverse transform are defined as follows :

$$
\begin{aligned}
L[x(t), g_0(t), g_1(t); s] &:= \int_0^\infty \frac{x(t)}{g_1(t)} e^{-\int_0^t \frac{s - g_0(\tau)}{g_1(\tau)} d\tau} dt \\
&= X(s)
\end{aligned}
\tag{18}
$$

$$
\begin{aligned}
L^{-1}[X(s), g_0(t), g_1(t); t] &= \frac{1}{2\pi i} \lim_{\omega_0 \to \infty} \int_{\sigma_0 - i\omega_0}^{\sigma_0 + i\omega_0} X(s) e^{\int_0^t \frac{s - g_0(\tau)}{g_1(\tau)} d\tau} ds \\
&= x(t)
\end{aligned}
\tag{19}
$$

where $s = \sigma + i\omega$ is a complex variable and σ_0 is chosen so that all the singular points of $X(s)$ lie to the left of the line $\mathrm{Re}\{s\} = \sigma_0$ in the complex plane $s = \sigma + i\omega$ [4]. If there arises no confusion, $L[x(t), g_0(t), g_1(t); s]$ and $L^{-1}[X(s), g_0(t), g_1(t); t]$ in Eq. (18) and Eq. (19), respectively, will be represented by $L[x(t)]$ and $L^{-1}[X(s)]$ for simplicity. If $g_0(t) = 0$ and $g_1(t) = 1$, the extended Laplace transform becomes the traditional Laplace transform.

Definition 3.2 : Extended exponential order[2][8].
For arbitrarily given functions $g_0(t)$ and $g_1(t)$ with $1/g_1(t) > 0$, a function $x(t)$ is said to be of extended exponential order with respect to $\{g_0(t), g_1(t)\}$ on $[0, \infty)$ or simply of extended exponential order on $[0, \infty)$, if there exist constants α and β such that

$$|x(t)| \le \beta e^{\int_0^t \frac{\alpha - g_0(\tau)}{g_1(\tau)} d\tau} \tag{20}$$

for all $t > 0$.

We now will prove the existence of the extended Laplace transform.

Lemma 3.1: If $x_1(t) \ge x_2(t)$ on some interval $[a, b]$ with $b > a$, then $G_{[a,b]}^{-1}x_1(t) \ge G_{[a,b]}^{-1}x_2(t)$.

Proof : Since $x_3(t) = x_1(t) - x_2(t)$ is non-negative on interval $[a, b]$ by the assumption, it follows from (2) that $G_{[a,b]}^{-1}x_3(t)$ is always non-negative. Now, from the linearity of the inverse operator $G_{[a,b]}^{-1}$, it follows that $G_{[a,b]}^{-1}x_1(t) \ge G_{[a,b]}^{-1}x_2(t)$. QED.

Lemma 3.2 : Let $x(t)$ be continuous on $[0, \infty)$ and $Gx(t)$ be of extended exponential order for some functions $\{ g_1(t), g_0(t) \}$. Then $x(t)$ is of extended exponential order on $[0, \infty)$.

Proof : By the assumption, $Gx(t)$ is of extended exponential order on $[0, \infty)$. That is, there exist some constants α and β such that

$$-\beta e^{\int_0^t \frac{\alpha - g_0}{g_1} d\tau} \le Gx(t) \le \beta e^{\int_0^t \frac{\alpha - g_0}{g_1} d\tau} \quad \text{for all } t > 0$$

where $Gx(t)$ is defined. Apply the inverse operator $G_{[0,t]}^{-1}$ to each term of the above inequality. By Lemma 3.1, the directions of inequalities are unchanged. Hence

$$\begin{aligned} G_{[0,t]}^{-1}(\beta e^{\int_0^t \frac{\alpha - g_0}{g_1} d\tau}) &= e^{-\int_0^t \frac{g_0}{g_1} d\tau} \int_0^t \frac{\beta e^{\int_0^\tau \frac{\alpha - g_0(\xi)}{g_1(\xi)} d\xi}}{g_1(\tau)} e^{\int_0^\tau \frac{g_0(\xi)}{g_1(\xi)} d\xi} d\tau \\ &= e^{-\int_0^t \frac{g_0}{g_1} d\tau} \int_0^t \frac{\beta}{g_1(\tau)} e^{\int_0^\tau \frac{\alpha}{g_1(\xi)} d\xi} d\tau \end{aligned}$$

If we let $\theta(\tau) = \int_0^\tau 1/g_1(\xi) d\xi$.

$$\begin{aligned} G_{[0,t]}^{-1}(\beta e^{\int_0^t \frac{\alpha - g_0}{g_1} d\tau}) &= e^{-\int_0^t \frac{g_0}{g_1} d\tau} \int_0^{\theta(t)} \beta e^{\alpha\theta} d\theta \\ &= \frac{\beta}{\alpha}\left(e^{\int_0^t \frac{\alpha - g_0}{g_1} d\tau} - e^{-\int_0^t \frac{g_0}{g_1} d\tau}\right) \end{aligned}$$

Also, it follows from Theorem 2.2 that $G_{[0,t]}^{-1}Gx(t) = x(t) - x(0)\exp(-\int_0^t g_0/g_1 d\tau)$. Rearranging the above results, we find :

$$\begin{aligned} -\frac{\beta}{\alpha}\left(e^{\int_0^t \frac{\alpha - g_0}{g_1} d\tau} - e^{-\int_0^t \frac{g_0}{g_1} d\tau}\right) &\le x(t) - x(0) e^{-\int_0^t \frac{g_0}{g_1} d\tau} \\ &\le \frac{\beta}{\alpha}\left(e^{\int_0^t \frac{\alpha - g_0}{g_1} d\tau} - e^{-\int_0^t \frac{g_0}{g_1} d\tau}\right) \end{aligned}$$

For $x(0) \ge 0$

$$\begin{aligned} |x(t)| &\le \frac{\beta}{\alpha} e^{\int_0^t \frac{\alpha - g_0}{g_1} d\tau} - \frac{\beta}{\alpha} e^{-\int_0^t \frac{g_0}{g_1} d\tau} + x(0) e^{-\int_0^t \frac{g_0}{g_1} d\tau} \\ &\le \beta_1 e^{\int_0^t \frac{\alpha_1 - g_0}{g_1} d\tau} \qquad \text{for all } t > 0 \end{aligned}$$

while for $x(0) < 0$

$$\begin{aligned} |x(t)| &\le \frac{\beta}{\alpha} e^{\int_0^t \frac{\alpha - g_0}{g_1} d\tau} - \frac{\beta}{\alpha} e^{-\int_0^t \frac{g_0}{g_1} d\tau} - x(0) e^{-\int_0^t \frac{g_0}{g_1} d\tau} \\ &\le \beta_2 e^{\int_0^t \frac{\alpha_2 - g_0}{g_1} d\tau} \qquad \text{for all } t > 0 \end{aligned}$$

Therefore, we can find α_3 and β_3 such that

$$|x(t)| \le \beta_3 e^{\int_0^t \frac{\alpha_3 - g}{f} d\tau} \quad \text{for all } t > 0$$

where, $\alpha_3 = \max(\alpha_1, \alpha_2)$, $\beta_3 = \max(\beta_1, \beta_2)$. Therefore, it follows that $x(t)$ is of extended exponential order on $[0, \infty)$ QED.

Theorem 3.1 : Existence of Extended Laplace transform. Let $x(t)$ be a piecewise continuous function and of extended exponential order w.r.t some functions $g_1(t)$ and $g_0(t)$ on $[0, \infty)$. Then $L[x(t)]$ exists. Also, there exists a constant α such that

$$\int_0^\infty \frac{x(t)}{g_1(t)} e^{-\int_0^t \frac{s - g_0(\tau)}{g_1(\tau)} d\tau} dt$$

converges for all $\mathrm{Re}\{s\} > \alpha$.

Proof : It is recalled that if $|u(t)| \le v(t)$ for all $t \ge a$, then $\int_a^\infty u(t)dt$ exists whenever $\int_a^\infty v(t)dt$ exists. Let α and β be chosen such that $|x(t)| \le \beta \exp(\int_0^t (\alpha - g_0)/g_1 \, d\tau)$ for all $t > 0$ because $x(t)$ is extended exponential order. Then

$$\begin{aligned} J &= \int_0^\infty \frac{\beta e^{\int_0^t \frac{\alpha - g_0(\tau)}{g_1(\tau)} d\tau}}{g_1(t)} e^{-\int_0^t \frac{s - g_0(\tau)}{g_1(\tau)} d\tau} dt \\ &= \beta \int_0^\infty \frac{1}{g_1(t)} e^{-\int_0^t \frac{s - \alpha}{g_1(\tau)} d\tau} dt \end{aligned}$$

If we let $\theta = \int_0^t 1/g_1 \, d\tau$, then

$$J = \lim_{t_0 \to \infty} \frac{\beta}{s - \alpha}[1 - e^{-(s-\alpha)\theta(t_0)}]$$

Also, let $\sigma_0 = \mathrm{Re}\{s\} - \alpha$ and $\rho = \lim_{t_0 \to \infty} \theta(t_0)$

$$J = \begin{cases} \dfrac{\beta}{s - \alpha}(1 - e^{-(s-\alpha)\rho}) & \text{if } \rho < \infty \\ \dfrac{\beta}{s - \alpha} & \text{if } \rho = \infty \text{ and } \sigma_0 > 0 \\ \pm\infty & \text{if } \rho = \infty \text{ and } \sigma_0 < 0 \\ \textit{undefined} & \text{if } \rho = \infty \text{ and } \sigma_0 = 0 \end{cases}$$

Therefore the comparison test implies that $L[x(t)]$ exists for all $\mathrm{Re}\{s\} > \alpha$. QED.

We now present several properties of operator G for the extended Laplace transform.

Theorem 3.2 : Let $x(t)$ be a continuous function on $(0, \infty)$ such that $Gx(t)$ is piecewise continuous function of extended exponential order on $[0, \infty)$. Then $L[x(t)]$ and $L[Gx(t)]$ exist, and

$$L[Gx(t)] = sL[x(t)] - x(0) \tag{21}$$

More generally, if $x(t), Gx(t), \cdots, G^{n-1}x(t)$ are continuous and $G^n x(t)$ is piecewise continuous and of extended exponential order on $[0, \infty)$, then

$$\begin{aligned} L[G^2 x(t)] &= s^2 X(s) - \lim_{t \to 0}[sx(t) + Gx(t)] \\ &= s^2 X(s) - sx(0) - Gx(0) \end{aligned} \tag{22}$$

$$\vdots$$

$$\begin{aligned} L[G^n x(t)] &= s^n X(s) - \lim_{t \to 0}[s^{n-1}x(t) + s^{n-2}Gx(t) + \cdots + G^{n-1}x(t)] \\ &= s^n X(s) - s^{n-1}x(0) - s^{n-2}Gx(0) - \cdots - G^{n-1}x(0) \end{aligned} \tag{23}$$

where $G^k x(0) = \lim_{t \to 0}[G^k x(t)]$, $k = 1, 2, \cdots, n-1$.

Proof : By Theorem 3.1, $L[x(t)]$ exists when $Gx(t)$ is piecewise continuous and of extended exponential order. To establish (21), we integrate by parts as follows

$$\begin{aligned} L[Gx(t)] &= \int_0^\infty \frac{g_1(t)\dot{x}(t) + g_0(t)x(t)}{g_1(t)} e^{-\int_0^t \frac{s - g_0(\tau)}{g_1(\tau)} d\tau} dt \\ &= \int_0^\infty \dot{x}(t) e^{-\int_0^t \frac{s - g_0(\tau)}{g_1(\tau)} d\tau} dt + \int_0^\infty \frac{g_0(t)}{g_1(t)} x(t) e^{-\int_0^t \frac{s - g_0(\tau)}{g_1(\tau)} d\tau} dt \end{aligned}$$

$$= x(t)e^{-\int_0^t \frac{s-g_0(\tau)}{g_1(\tau)}d\tau}\Big|_0^{\infty} + s\int_0^{\infty} \frac{x}{g_1(t)} e^{-\int_0^t \frac{s-g_0(\tau)}{g_1(\tau)}d\tau} d\tau$$
$$= sX(s) - x(0)$$

In the third line of the above derivation, we find that

$$\lim_{t\to\infty} |x(t)e^{-\int_0^t \frac{s-g_0}{g_1}d\tau}| \le \lim_{t\to\infty} \beta e^{\int_0^t \frac{\alpha-g_0}{g_1}d\tau} e^{-\int_0^t \frac{\sigma-g_0}{g_1}d\tau}$$
$$= \begin{cases} 0 & \text{if } \mathrm{Re}\{s\} > \alpha \text{ and } \lim_{t\to\infty}\int_0^t \frac{1}{g_1}d\tau = \infty \\ \beta e^{(\alpha-\sigma)\rho} & \text{if } \lim_{t\to\infty}\int_0^t \frac{1}{g_1}d\tau = \rho \end{cases}$$

since $x(t)$ is of extended exponential order. Therefore, when Re{s} is sufficiently large, we can obtain Eq. (21) with the assumption of $\lim_{t\to\infty}\int_0^t 1/g_1 d\tau = \infty$. Finally Eqs. (22) and (23) can be established by repeating the result (21). QED.

Theorem 3.3 : If $x(t)$ is piecewise continuous and of extended exponential order on $[0,\infty)$, then so is $G_{[0,t]}^{-1}x(t)$, and

$$L[G_{[0,t]}^{-1}x(t)] = L[e^{-\int_0^t \frac{g_0}{g_1}d\tau} \int_0^t \frac{x(t)}{g_1} e^{\int_0^t \frac{g_0}{g_1}d\tau} dt]$$
$$= \frac{1}{s}L[x(t)] \tag{24}$$

More generally,

$$L[\underbrace{G_{[0,t]}^{-1}\cdots G_{[0,t]}^{-1}}_{\text{n times}}x(t)] = \frac{1}{s^n}L[x(t)] \tag{25}$$

Proof : By Lemma 3.2, $G_{[0,t]}^{-1}x(t)$ is piecewise continuous and of extended exponential order on $[0,\infty)$. To prove the basic formula, we use integration by parts again as follows:

$$L[G_{[0,t]}^{-1}x(t)] = L[e^{-\int_0^t \frac{g_0}{g_1}d\tau} \int_0^t \frac{x(t)}{g_1} e^{\int_0^t \frac{g_0}{g_1}d\tau} dt]$$
$$= \int_0^{\infty} \frac{1}{g_1}[e^{-\int_0^t \frac{g_0}{g_1}d\tau} \int_0^t \frac{x(u)}{g_1} e^{\int_\tau^u \frac{g_0}{g_1}d\tau} du] e^{-\int_0^t \frac{s-g_0}{g_1}d\tau} dt$$
$$= -\frac{1}{s}\int_0^t \frac{x(u)}{g_1} e^{\int_0^u \frac{g_0}{g_1}d\tau} du\, e^{-\int_0^t \frac{s}{g_1}d\tau}\Big|_0^{\infty} + \frac{1}{s}\int_0^{\infty} \frac{x(t)}{g_1} e^{-\int_0^t \frac{s-g_0}{g_1}d\tau} dt$$
$$= \frac{1}{s}X(s)$$

In the third line, when Re{s} is sufficiently large

$$\lim_{t\to\infty} e^{-\int_0^t \frac{s}{g_1}d\tau} \int_0^t \frac{x(u)}{g_1} e^{\int_0^u \frac{g_0}{g_1}d\tau} du = 0$$

because $x(t)$ is extended exponential order. Finally, (25) is established by iterating this result. QED.

Theorem 3.4 : Complex shift.

If

$$L[x(t)] = X(s)$$

then

$$L[e^{\mp\int_0^t \frac{a}{g_1}d\tau} x(t)] = X(s \pm a) \tag{26}$$

Proof : By definition,

$$L[e^{\mp\int_0^t \frac{a}{g_1}d\tau} x(t)] = \int_0^{\infty} \frac{1}{g_1} e^{\mp\int_0^t \frac{a}{g_1}d\tau} x(t) e^{-\int_0^t \frac{s-g_0}{g_1}d\tau} dt$$
$$= \int_0^{\infty} \frac{1}{g_1} x(t) e^{-\int_0^t \frac{(s\pm a)-g_0}{g_1}d\tau} dt$$
$$= X(s \pm a)$$

QED.

Next we shall consider the convolution property of LQTI systems [6][7]. Define the extended convolution $\otimes$ of two time functions as follows :

$$y(t) = h(t) \otimes u(t)$$
$$=: \int_0^t \frac{1}{g_1(\tau)} h(t,\tau) u(\tau) d\tau$$
$$= \int_0^t \frac{1}{g_1(\tau)} h(\int_\tau^t \frac{1}{g_1(\xi)} d\xi) e^{-\int_\tau^t \frac{g_0(\xi)}{g(\xi)}d\xi} u(\tau) d\tau \tag{27}$$

where,

$$h(t,\tau) = h(\int_\tau^t \frac{1}{g_1(\xi)} d\xi) e^{-\int_\tau^t \frac{g_0(\xi)}{g_1(\xi)}d\xi}$$

We see that Eq. (27) is equal to the traditional LTI convolution if $g_1(t) = 1$ and $g_0(t) = 0$.

Theorem 3.5 : Extended Convolution Theorem.

Suppose that $\lim_{t\to\infty} v(t) = \int_0^t 1/g_1(\xi) d\xi = \infty$. Consider the convolution of two functions $h(t)$ and $u(t)$ as in Eq. (27). Then, taking the extended Laplace transform of both sides on Eq. (27), we have

$$Y(s) = H(s)U(s) \tag{28}$$

where,

$$Y(s) = L[y(t), g_0(t), g_1(t); s]$$
$$= L[h(t) \otimes u(t), g_0(t), g_1(t); s]$$

while

$$H(s) = L[h(t,0), g_0(t), g_0(t); s],\ U(s) = L[u(t), g_0(t), g_1(t); s].$$

Proof : From Eq. (27), we find that

$L[y(t)]$
$$= \int_0^{\infty} (\int_0^t \frac{1}{g_1(\tau)} h(\int_\tau^t \frac{1}{g_1(\xi)} d\xi) e^{-\int_\tau^t \frac{g_0}{g_1}d\xi} u(\tau) d\tau) \frac{1}{g_1(t)} e^{-\int_0^t \frac{s-g_0(\xi)}{g_1(\xi)}d\xi} dt,$$
$$= \int_0^{\infty} \int_\tau^{\infty} \frac{1}{g_1(t)g_1(\tau)} h(\int_\tau^t \frac{1}{g_1(\xi)} d\xi) e^{-\int_\tau^t \frac{g_0}{g_1}d\xi} u(\tau) e^{-\int_0^t \frac{s-g_0(\xi)}{g_1(\xi)}d\xi} dt d\tau$$

where the integration in the first line is performed over the first eighth region of the $t\tau$ - plane represented by the inequalities

$$0 \le \tau \le t \quad \text{and} \quad 0 \le t < \infty$$

See Fig. 3.1. But this region can be represented by

$$0 \le \tau < \infty \quad \text{and} \quad \tau \le t < \infty$$

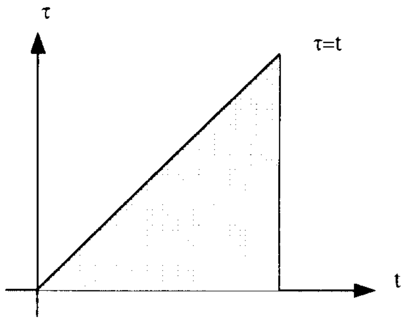

Fig. 3.1 Region of integration

Therefore, the double integral in the first line can be written as that of the second line.

Let

$$\underbrace{\int_\tau^t \frac{1}{g_1} d\xi}_{= w} = \int_\tau^0 \frac{1}{g_1} d\xi + \int_0^t \frac{1}{g_1} d\xi$$

Then,

$$dw = \frac{1}{g_1}dt \,,\ \int_0^t \frac{1}{g_1}d\xi = w + \int_0^\tau \frac{1}{g_1}d\xi$$

Hence,

$$\begin{aligned}
&L[y(t)] \\
&= \int_0^\infty \int_0^\infty \frac{1}{g_1(\tau)} h(w) e^{-\left(\int_\tau^0 \frac{g_0(\xi)}{g_1(\xi)}d\xi + \int_0^t \frac{g_0(\xi)}{g_1(\xi)}d\xi\right)} u(\tau) e^{-\int_0^t \frac{s-g_0(\xi)}{g_1(\xi)}d\xi} dw d\tau \\
&= \int_0^\infty h(w) e^{-sw} dw \int_0^\infty \frac{1}{g_1(\tau)} u(\tau) e^{-\int_0^\tau \frac{s-g_0(\xi)}{g_1(\xi)}d\xi} d\tau \\
&= H(s)U(s)
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
L[h(t,0)] &= \int_0^\infty \frac{1}{g_1(t)} h\left(\int_0^t \frac{1}{g_1(\xi)}d\xi\right) e^{-\int_0^t \frac{g_0(\xi)}{g_1(\xi)}d\xi} e^{-\int_0^t \frac{s-g_0(\xi)}{g_1(\xi)}d\xi} dt \\
&= \int_0^\infty \frac{1}{g_1(t)} h\left(\int_0^t \frac{1}{g_1(\xi)}d\xi\right) e^{-\int_0^t \frac{s}{g_1(\xi)}d\xi} dt \qquad (29) \\
&= \int_0^\infty h(v) e^{-sv} dv \\
&= H(s)
\end{aligned}$$

QED.

4. APPLICATION

In the traditional LTI circuit theory, some analysis and design of LTI filter are based on the concepts of eigenvalue and eigenfunction in relation to the differential operator D. In the LQTI system theory, there is also an eigenstructure explained in terms of eigenvalue and eigenfunction corresponding to the operator G. Therefore, there is a very simple method to analyze and design of LQTI filters as in case of LTI filters. To be specific, we consider as an example a physical circuit including a varactor diode as a time varying capacitor $c(t)$ and a voltage controlled resistor $r(t)$ as a time varying resistor as shown in Fig 4.1. For small signal analysis, i-v relations of the devices $c(t)$ and $r(t)$ are modeled as

$$i_c = c(t)\frac{dv}{dt} + \frac{dc}{dt}v,$$
$$i_r = \frac{1}{r(t)}v$$

Consider now the LQTI system shown in Fig 4.1.

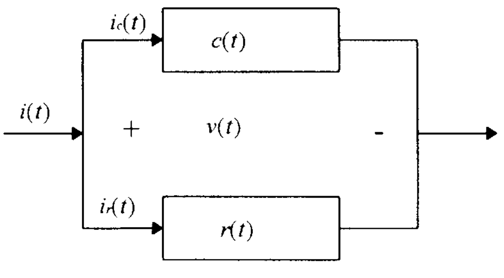

Fig. 4.1 LQTI elementary circuit G .

Then, it follows that

$$\begin{aligned}
i &= i_r + i_c \\
&= c(t)\frac{dv}{dt} + \left(\frac{1}{r(t)} + \frac{dc(t)}{dt}\right)v \qquad (30) \\
&= g_1(t)\frac{dv}{dt} + g_0(t)v
\end{aligned}$$

That is, $i = Gv$ where $G = c(t)D + (1/r(t) + Dc(t))$. It is a LQTI circuit system. In this system, $g_1(t)$ and $g_0(t)$ can be changed independently by the bias and control voltages, from the large signal point of view. By the same method, using a time varying inductor and resistor, we can make the inverse of the operator G. We now consider a circuit consisting of devices which can be represented by G.

Example 4.1 : LQTI circuit.

Consider the LQTI circuit shown in Fig 4.2. R_1, R_2, C_1, C_2 are constants. In the figure, $G_1 = C_1G = C_1(g_1D + g_0)$, $G_2 = C_2G = C_2(g_1D + g_0)$ with C_1,C_2 being constants.

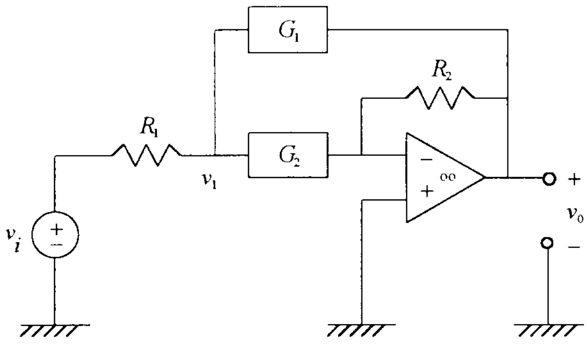

Fig. 4.2 An example of LQTI circuit.

To perform the small signal analysis, recall that the operator G is linear. By the node analysis, we can obtain the following relations :

$$\frac{1}{R_1}(v_1 - v_i) + C_2Gv_1 + C_1G(v_1 - v_0) = 0$$
$$C_2Gv_1 + \frac{1}{R_2}v_0 = 0$$

By combining, we obtain

$$-\frac{v_0}{R_1R_2} - \frac{1}{R_2}(C_1 + C_2)Gv_0 - C_1^2G^2v_0 = \frac{1}{R_1}C_1Gv_i \qquad (31)$$

Let us take the extended Laplace transform of both sides of Eq. (31). Then,

$$\frac{V_0(s)}{V_i(s)} = -\frac{\dfrac{s}{R_1C_1}}{s^2 + \dfrac{1}{R_2}\left(\dfrac{1}{C_1} + \dfrac{C_2}{C_1^2}\right)s + \dfrac{1}{R_1R_2C_1^2}} \qquad (32)$$

This circuit is a second order band pass filter in view point of the sinusoidal signal modulated by $g_1(t)$, $g_0(t)$ or operator G . But, the circuit may not be band pass filter when considering in view point of traditional Laplace transform, according to the functions $g_1(t)$, $g_0(t)$. The physical meaning of complex variable s is different from that of the traditional Laplace transform. In other words, the complex variable s represents a function modulated by $g_1(t)$, $g_0(t)$ in time domain. We can analyze and design the LQTI circuit in this manner.

Although the Kalman filter is a very efficient filtering technique, there is no modulation (frequency shift) concept. On the other hand, there is a modulation concept in Wiener filter, which is a LTI filter. Therefore, the Wiener filters and modulation property is used in the present communication system for the effective frequency utilization. From this point of view, we may comment that LQTI filter and its related systems can be used for more efficient communication and frequency resource utilization.

5. CONCLUDING REMARKS

The eigenstructure of the LQTI system is studied along with extended Laplace transform and its various properties. Based on the eigenstructure of LQTI system, it is possible to develop various useful methodologies of analysis and design for LQTI systems as the eigenstructure of LTI systems induces such elegant methods in the linear circuit theory, in automatic control, modulation in communication and spectral analysis of signals. Also it is shown that the LQTI filter can be analyzed and designed when the concept of eigenstructure of LQTI system is properly utilized.

In this paper, we have introduced the operator G composed of a first order derivative and two arbitrary functions. This concept can be easily generalized. That is, we can define the operator G as $G = g_n D^n + g_{n-1} D^{n-1} + \cdots + g_2 D^2 + g_1 D^1 + g_0$, and from $Gx = \lambda x$, we can develop the theory in a similar manner. In the case of defining the Extended Laplace transform for n-dimensional operator G, however, we must construct the n-dimensional partial differential equation. It is finally noted that the proposed theory for a class of time varying linear systems can have various applications as indicated in section IV and in the robust control of linear time varying systems. Also, it is commented that there can be some time varying linear differential equations which cannot be represented by the linear combination of operator G, although any second order linear time varying differential equation can be represented by the operator G completely.

APPENDIX A.
Proof of Theorem 2.2

Firstly, assume that system matrix A of Eq. (7) has the distinct eigenvalues. Then,

$$\begin{gathered} g_1 D y_1 + g_0 y_1 = \lambda_1 y_1 \\ g_1 D y_2 + g_0 y_2 = \lambda_2 y_2 \\ \vdots \\ g_1 D y_n + g_0 y_n = \lambda_n y_n \end{gathered}$$

This implies :

$$y_i(t) = e^{\int_0^t \frac{\lambda_i - g_0}{g_1} d\tau} y_i(t_0), \quad i = 1, 2, \cdots, n$$

That is,

$$\mathrm{y}(t) = Q(t)\mathrm{y}(t_0)$$

where

$$\mathrm{y} = [y_1, \cdots, y_n]^t$$

Since $GP\mathbf{x} = PG\mathbf{x}$, Eq. (8) is obtained from the nonsingular transformation $\mathbf{x} = P\mathrm{y}$ and initial conditions.

Secondly, when the Eq. (7) has multiple eigenvalues, it is obvious that there exists a nonsingular transform P such that $P^{-1}AP$ is a Jordan canonical form matrix. Consider the ith block of $P^{-1}AP$ with eigenvalue λ_i with multiplicity m_i. Then

$$g_1 D \begin{bmatrix} y_i \\ y_{i+1} \\ \vdots \\ y_{i+m_i-2} \\ y_{i+m_i-1} \end{bmatrix} + g_0 \begin{bmatrix} y_i \\ y_{i+1} \\ \vdots \\ y_{i+m_i-2} \\ y_{i+m_i-1} \end{bmatrix} = \begin{bmatrix} \lambda_i & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_i & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_i & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda_i \end{bmatrix} \begin{bmatrix} y_i \\ y_{i+1} \\ \vdots \\ y_{i+m_i-2} \\ y_{i+m_i-1} \end{bmatrix}$$

By solving the above equations, we obtain :

$$y_{i+m_i-1}(t) = e^{\int_0^t \frac{\lambda_i - g_0}{g_1} d\tau} y_{i+m_i-1}(t_0)$$

$$y_{i+m_i-2}(t) = e^{\int_0^t \frac{\lambda_i - g_0}{g_1} d\tau} \left(y_{i+m_i-2}(t_0) + y_{i+m_i-1}(t_0) \int_0^t \frac{1}{g_1} dt_1 \right)$$

$$y_{i+m_i-3}(t) = e^{\int_0^t \frac{\lambda_i - g_0}{g_1} d\tau} \left(y_{i+m_i-3}(t_0) + y_{i+m_i-2}(t_0) \int_0^t \frac{1}{g_1} dt_1 + y_{i+m_i-1}(t_0) \int_0^t \frac{1}{g_1} \int_0^{t_1} \frac{1}{g_1} dt_2 dt_1 \right)$$

$$\vdots$$

$$y_i(t) = e^{\int_0^t \frac{\lambda_i - g_0}{g_1} d\tau} \left(y_i(t_0) + y_{i+1}(t_0) \int_0^t \frac{1}{g_1} dt_1 + \cdots + y_{i+m_i-1}(t_0) \int_0^t \frac{1}{g_1} \int_0^{t_1} \frac{1}{g_1} \cdots \int_0^{t_{m_i-2}} \frac{1}{g_1} dt_{m_i-1} \cdots dt_2 dt_1 \right)$$

It now follows from Lemma A.1 stated below that the following relation holds :

$$M_i(t) = S_i(t) Q_i(t) M_i(t_0)$$

where

$$M_i(t) = [y_i(t)\, y_{i+1}(t) \cdots y_{i+m_i-2}(t)\, y_{i+m_i-1}(t)]^t$$

Therefore we can obtain Eq. (10) from the transformation $\mathbf{x} = P\mathrm{y}$ and initial conditions. QED.

Lemma A.1 [5] Let $\beta(t)$ be a integrable function. Then we can obtain the following relation.

$$\underbrace{\int_0^t \beta \int_0^{t_1} \beta \cdots \int_0^{t_{k-1}} \beta}_{k \text{ times}} dt_k \cdots dt_2 dt_1 = \frac{\left(\int_0^t \beta dt_1\right)^k}{k!}$$

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