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DESIGN OF ROBUST GAIN SCHEDULING CONTROLLERS IN UNCERTAIN NONLINEAR SYSTEMS

°Seon-Ho Lee* and Jong-Tae Lim*

* Department of Electrical Engineering, Korea Advanced Institute of Science and Technology
373-1 Kusong-dong, Yusong-gu, Taejon, Korea
Tel : +82-42-869-5441 Fax : +82-42-869-3410 E-mail : shlee@stcon2.kaist.ac.kr

Abstracts This paper considers the output regulation problems on uncertain systems. Using NR-estimator(on-line), a family of equilibrium points for the uncertain system is computed. The state variables of the closed loop system track the average value of the obtained equilibrium manifold by dynamic state feedback control.

Keywords Equilibrium Manifold, Average Equilibrium Point, NR-estimator, Dynamic State Feedback

1. INTRODUCTION

Recently there has been considerable progress in the theory of gain scheduling[1] [2] [3] [4] [5]. First, the control law for the systems with slowly varying parameters has been developed. Huang and Rugh considered nonlinear systems with both measured and unmeasured disturbance signals, and proposed output feedback control law for reference tracking problem[1]. Analytical framework for the state feedback control law was also proposed for regulation problem[2]. Moreover, state feedback control laws with the derivative information on the scheduling variables are proposed for somewhat fast time varying inputs[6] [7]. However, these developed control schemes are restricted to the exactly known plant dynamics. [8] and [9] proposed a dynamic state feedback controller using internal model principle. They tried to achieve asymptotic tracking and disturbance rejection for uncertain nonlinear systems.

In this paper, NR-estimator computes a family of equilibrium points(equilibrium manifold) for an uncertain system with information about the system uncertainties on line. Moreover, a dynamic static feedback controller is proposed so that the state variables asymptotically track the average of the equilibrium manifold with certain error bound. Owing to the output feedback compensator, some of the average equilibrium manifold are continuously adjusted to reduce the output error and the resulting overall system reveals robust property for the uncertainties.

2. PROBLEM DESCRIPTION

We note that for given $a \in R^n$ and $A \in R^{n\times m}$, $(a)_i$ denotes the ith element of a and $(A)_{ij}$ the (i,j)th element of A.

We consider the following uncertain system

$$\begin{aligned} \dot{x} &= f(x,w) + \Delta f(x,w) + (G(x,w) + \Delta G(x,w))u \\ y &= h(x,u,w),\ x(0) = x_o,\ t \geq 0 \end{aligned} \qquad (1)$$

where $x \in R^n$ is the state vector, $w \in R^m$ is the exogenous signal, $u \in R^p$ is the control input, and $y \in R^p$ is the output. The uncertainties Δf and ΔG are unknown and bounded. The functions $f : R^n \times R^m \to R^n$, $G : R^n \times R^m \to R^{n\times p}$, and $h : R^n \times R^m \times R^p \to R^p$ are assumed to be continuously differentiable. The control objective is to minimize ϵ while rejecting the plant uncertainties such that

$$\lim_{t\to\infty} \|r_d - y\| < \epsilon \qquad (2)$$

where $r_d \in R^p$ is the constant reference input.

Assumption 1 There exists an open neighborhood $\Gamma \in R^m$ of the origin and a smooth function $(\mathbf{x}_0(w),\mathbf{u}_0(w))$ for $w \in \Gamma$ such that

$$\begin{aligned} 0 &= f(\mathbf{x}_0(w), w) + G(\mathbf{x}_0(w), w)\mathbf{u}_0(w) \\ r_d &= h(\mathbf{x}_0(w), \mathbf{u}_0(w), w) \end{aligned} \qquad (3)$$

Assumption 2 There exist finite functions $\underline{\alpha}(w,t)$, $\overline{\alpha}(w,t)$, $\underline{\beta}(w,t)$, and $\overline{\beta}(w,t)$ such that for $1 \leq i \leq n$ and $1 \leq j \leq p$,

$$\begin{aligned} &\underline{\alpha}(w,t) \leq |(\Delta f)_i| \leq \overline{\alpha}(w,t) \\ &\underline{\beta}(w,t) \leq |(\Delta G)_{ij}| \leq \overline{\beta}(w,t), \quad w \in \Gamma, t \leq 0 \end{aligned} \qquad (4)$$

For notational convenience, we let

$$\begin{aligned} F(x,u,w,\Delta f,\Delta G) &= \\ f(x,w) + \Delta f(x,w) + &\ (G(x,w) + \Delta G(x,w))u \end{aligned} \qquad (5)$$

Assumption 3 There exists a smooth function $(\tilde{\mathbf{x}}(w),\tilde{\mathbf{u}}(w))$ for $w \in \Gamma$ such that

$$\begin{aligned} 0 &= F(\tilde{\mathbf{x}}(w), \tilde{\mathbf{u}}(w), w, \Delta f(\tilde{\mathbf{x}}(w), \tilde{\mathbf{u}}(w)), \\ &\quad \Delta G(\tilde{\mathbf{x}}(w), \tilde{\mathbf{u}}(w))) \\ r_d &= h(\tilde{\mathbf{x}}(w), \tilde{\mathbf{u}}(w), w) \end{aligned} \qquad (6)$$

and $\tilde{\mathbf{x}}(w) = \mathbf{x}_0(w)$ and $\tilde{\mathbf{u}}(w) = \mathbf{u}_0(w)$ if $\Delta f = \Delta G = 0$.

Here, we notice that the solution in (3) is not unique, and nonzero Δf and ΔG displace the obtained $(\mathbf{x}_0, \mathbf{u}_0)$ to unknown $(\tilde{\mathbf{x}}, \tilde{\mathbf{u}})$.

Define

$$\begin{aligned} \Phi &= \{\phi \in R^n | (\phi)_i \text{ is } \underline{\alpha} \text{ or } \overline{\alpha}, 1 \leq i \leq n\} \\ \Theta &= \{\theta \in R^{n\times p} | (\theta)_{ij} \text{ is } \underline{\beta} \text{ or } \overline{\beta}, 1 \leq i \leq n, \\ &\quad 1 \leq j \leq p\} \end{aligned} \qquad (7)$$

If we consider $(\mathbf{x},\mathbf{u})$ near $(\tilde{\mathbf{x}},\tilde{\mathbf{u}})$, i.e., $[\tilde{\mathbf{x}}^T,\tilde{\mathbf{u}}^T]^T = [(\mathbf{x}+\Delta\mathbf{x})^T,(\mathbf{u}+\Delta\mathbf{u})^T]^T$ with sufficiently small $[\Delta\mathbf{x}^T,\Delta\mathbf{u}^T]^T$, then applying Newton-Raphson method we find an approximate increment $[d\mathbf{x}^T,d\mathbf{u}^T]^T$ such that

$$\begin{bmatrix} F(\mathbf{x},\mathbf{u},w,\Delta f,\Delta G) \\ h(\mathbf{x},\mathbf{u},w,\Delta f,\Delta G)-r_d \end{bmatrix} + \begin{bmatrix} \frac{\partial F(\mathbf{x},\mathbf{u},w,\Delta f,\Delta G)}{\partial x} & \frac{\partial F(\mathbf{x},\mathbf{u},w,\Delta f,\Delta G)}{\partial u} \\ \frac{\partial h(\mathbf{x},\mathbf{u},w)}{\partial x} & \frac{\partial h(\mathbf{x},\mathbf{u},w)}{\partial u} \end{bmatrix} \begin{bmatrix} d\mathbf{x} \\ d\mathbf{u} \end{bmatrix} = 0 \tag{8}$$

with an assumption that the jacobian matrix is invertible. Here, we propose a iterative law to obtain $[\tilde{\mathbf{x}}^T,\tilde{\mathbf{u}}^T]^T$ such that

$$\begin{bmatrix} \mathbf{x}_{k+1} \\ \mathbf{u}_{k+1} \end{bmatrix} = \begin{bmatrix} \mathbf{x}_k + d\mathbf{x}_k \\ \mathbf{u}_k + d\mathbf{u}_k \end{bmatrix} \tag{9}$$

$$\begin{bmatrix} d\mathbf{x}_k \\ d\mathbf{u}_k \end{bmatrix} = -\begin{bmatrix} \frac{F_k(\cdot)-F_{k-1}(\cdot)}{\mathbf{x}_k-\mathbf{x}_{k-1}} & \frac{F_k(\cdot)-F_{k-1}(\cdot)}{\mathbf{u}_k-\mathbf{u}_{k-1}} \\ \frac{h_k(\cdot)-h_{k-1}(\cdot)}{\mathbf{x}_k-\mathbf{x}_{k-1}} & \frac{h_k(\cdot)-h_{k-1}(\cdot)}{\mathbf{u}_k-\mathbf{u}_{k-1}} \end{bmatrix}^{-1} \begin{bmatrix} F_k(\cdot) \\ h_k(\cdot)-r_d \end{bmatrix} \tag{10}$$

where $F_k(\cdot) = F(\mathbf{x}_k,\mathbf{u}_k,w,\Delta f,\Delta G)$ and $h_k(\cdot) = h(\mathbf{x}_k,\mathbf{u}_k,w)$ with the initial value $[\mathbf{x}_0{}^T,\mathbf{u}_0{}^T]^T$ from assumption 1. However, we can't compute the equilibrium point $(\tilde{\mathbf{x}},\tilde{\mathbf{u}})$ due to the unknown Δf and ΔG. Thus, using assumption 2, we obtain approximately estimated equilibrium point $(\tilde{\mathbf{x}}_k,\tilde{\mathbf{u}}_k)$ shown in Fig 1. First, we regard all the solutions of (10) with $\Delta f = \phi \in \Phi$ and $\Delta G = \theta \in \Theta$ as equilibrium manifold $\{\mathbf{x}_k,\mathbf{u}_k\}$ with assumption of the existence of the finite solution $\mathbf{x}_k,\mathbf{u}_k$ for slowly varying $w(t)$. We denote the average value of $\{\mathbf{x}_k,\mathbf{u}_k\}$ as the average equilibrium point $(\tilde{\mathbf{x}}_k,\tilde{\mathbf{u}}_k)$. This scheme is executed by NR-estimator at every step k. Overall system is depicted in Fig. 2.
Letting $e = x - \tilde{x}$, then $||e|| \leq ||x-\tilde{\mathbf{x}}_k|| + ||\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}||$. Here, $||x-\tilde{\mathbf{x}}_k||$ is determined by the performance of the state feedback controller and $||\tilde{\mathbf{x}}_k-\tilde{\mathbf{x}}||$ by the output of NR-estimator with the estimated bound $\underline{\alpha},\overline{\alpha},\underline{\beta}$, and $\overline{\beta}$.

Theorem 1 There exists $d(t)$ such that

$$||\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}|| \leq d(t)/2 \tag{11}$$

Proof: Suppose that for $\phi \in \Phi$ and $\theta \in \Theta$, $w \in \Gamma$, NR-estimator gives solution $(\mathbf{x}_k,\mathbf{u}_k)$ around $(\tilde{\mathbf{x}},\tilde{\mathbf{u}})$ such that

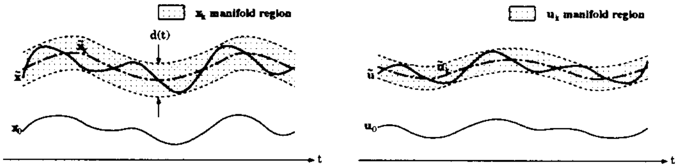
$$\begin{aligned} 0 &= F(\mathbf{x}_k,\mathbf{u}_k,w,\phi,\theta) \\ r_d &= h(\mathbf{x}_k,\mathbf{u}_k,w) \end{aligned} \tag{12}$$

when the initial value $(\mathbf{x}_0(w),\mathbf{u}_0(w))$.
Then, from Fig. 1-(a) we obtain

$$d(t) = [\sum_{i=1}^{n}\{\max_{\phi\in\Phi,\theta\in\Theta}(\mathbf{x}_k)_i - \min_{\phi\in\Phi,\theta\in\Theta}(\mathbf{x}_k)_i\}^2]^{1/2} \tag{13}$$

The control law which guarantees the control objective is given by

$$\begin{aligned} u &= k(z,x) \\ \dot{z} &= l(z,y) \end{aligned} \tag{14}$$



(a) (b)

Fig. 1: Trajectory of the average equilibrium manifold $(\tilde{\mathbf{x}}_k,\tilde{\mathbf{u}}_k)$ in uncertain system

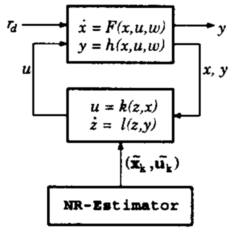


Fig. 2: Block diagram of the overall feedback system

where $z \in R^q$. The corresponding linearized closed loop system is written in the form

$$\begin{aligned} \dot{x}_\delta &= \tilde{A}(\mathbf{w},\Delta f,\Delta G)x_\delta + \tilde{B}(\mathbf{w},\Delta f,\Delta G)u_\delta \\ &\quad + \tilde{E}(\mathbf{w},\Delta f,\Delta G)w_\delta \\ y_\delta &= \tilde{C}(\mathbf{w})x_\delta + \tilde{D}(\mathbf{w})u_\delta + \tilde{F}(\mathbf{w})w_\delta \\ u_\delta &= K_1(\mathbf{w})z + K_2(\mathbf{w})x_\delta \\ \dot{z} &= L_1(\mathbf{w})z + L_2(\mathbf{w})y_\delta \end{aligned} \tag{15}$$

Here, the derivative variables are defined as $x_\delta = x - \tilde{\mathbf{x}}_k(\mathbf{w})$, $w_\delta = w - \mathbf{w}$, $u_\delta = u - \tilde{\mathbf{u}}_k(\mathbf{w})$, and $y_\delta = y - r_d$. The linearized system coefficients by $\tilde{A}(\mathbf{w},\Delta f,\Delta G) = \frac{\partial F(\tilde{\mathbf{x}}_k,\tilde{\mathbf{u}}_k,\mathbf{w},\Delta f,\Delta G)}{\partial x}$, $\tilde{B}(\mathbf{w},\Delta f,\Delta G) = \frac{\partial F(\tilde{\mathbf{x}}_k,\tilde{\mathbf{u}}_k,\mathbf{w},\Delta f,\Delta G)}{\partial u}$, $\tilde{E}(\mathbf{w},\Delta f,\Delta G) = \frac{\partial F(\tilde{\mathbf{x}}_k,\tilde{\mathbf{u}}_k,\mathbf{w},\Delta f,\Delta G)}{\partial w}$, $\tilde{C}(\mathbf{w}) = \frac{\partial h(\tilde{\mathbf{x}}_k,\tilde{\mathbf{u}}_k,\mathbf{w})}{\partial x}$, $\tilde{D}(\mathbf{w}) = \frac{\partial h(\tilde{\mathbf{x}}_k,\tilde{\mathbf{u}}_k,\mathbf{w})}{\partial u}$, and $\tilde{F}(\mathbf{w}) = \frac{\partial h(\tilde{\mathbf{x}}_k,\tilde{\mathbf{u}}_k,\mathbf{w})}{\partial w}$ and the linearized control law coefficients $K_1(\mathbf{w}),K_2(\mathbf{w}),L_1(\mathbf{w})$, and $L_2(\mathbf{w})$ design factors. In order to obtain the desired pole-placement in the linearized closed loop system

$$\begin{bmatrix} \tilde{A}(0,0,0) & 0 \\ L_2(0)\tilde{C}(0) & L_1(0) \end{bmatrix}, \begin{bmatrix} \tilde{B}(0,0,0) \\ L_2(0)\tilde{D}(0) \end{bmatrix} \tag{16}$$

should be controllable pair. Moreover,

$$\begin{bmatrix} \tilde{A}+\tilde{B}K_2 & \tilde{B}K_1 \\ L_2(\tilde{C}+\tilde{D}K_2) & L_1 + L_2\tilde{D}K_1 \end{bmatrix} \tag{17}$$

should be Hurwitz.

Theorem 2 Suppose (16) is controllable pair and the eigenvalues of (17) have real parts less than $\lambda < 0$ when $\Delta f = \Delta G = 0$. Then, given positive constants ρ and T there exist positive constants $\varepsilon(\rho)$, ε_1, ε_2, $\delta_1(\rho)$, and $\delta_2(\rho,T)$ for which following properties hold. If for a continuously differentiable $w(t) \in \Gamma, t \geq 0$

$$\begin{aligned} &||\Delta f|| < \varepsilon_1, \ \ ||\Delta G|| < \varepsilon_2 \\ &||x(0) - \tilde{\mathbf{x}}_k(w(0))|| < \delta_1 \\ &\tfrac{1}{T}\int_t^{t+T} ||\dot{w}(\sigma)||d\sigma < \delta_2, \ \ t \geq 0 \end{aligned} \tag{18}$$

then

$$||x(t) - \tilde{\mathbf{x}}_k(w(t))|| < \rho \tag{19}$$

and

$$\lim_{t\to\infty} \|r_d - y\| < \epsilon(\rho) \tag{20}$$

The proof is directly derived from [10]. Finally, we obtain

$$\begin{aligned} u &= \tilde{\mathbf{u}}_k + K_1(w)z + K_2(w)(x - \tilde{\mathbf{x}}_k) \\ \dot{z} &= L_1(w)z + L_2(w)(y - r_d) \end{aligned} \tag{21}$$

Letting $\tilde{\mathbf{u}}'_k = \tilde{\mathbf{u}}_k + K_1(w)z$, then

$$u = \tilde{\mathbf{u}}'_k + K_2(w)(x - \tilde{\mathbf{x}}_k) \tag{22}$$

We notice that (22) is the typical state feedback controller form in gain scheduling[2]. The equilibrium point is changed from $(\tilde{\mathbf{x}}_k, \tilde{\mathbf{u}}_k)$ to $(\tilde{\mathbf{x}}_k, \tilde{\mathbf{u}}'_k)$. In other words, $\tilde{\mathbf{u}}_k$ is modified into $\tilde{\mathbf{u}}'_k$ so as to reduce output error.

Example: Consider the Boost type DC to DC converter shown in Fig. 3. Under the assumption of fast

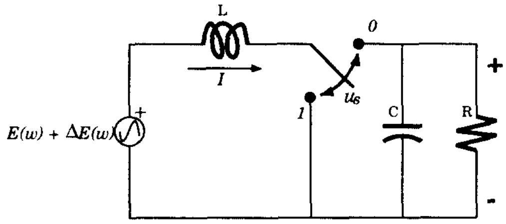


Fig. 3: Boost type DC to DC Converter

switching, we define an average model by formally replacing the discontinuous control function u_s by continuous smooth function u.

$$\begin{aligned} \dot{x}_1 &= -\tfrac{1}{L}x_2 + \tfrac{E(w)+\Delta E(w)}{L} + \tfrac{1}{L}x_2 u \\ \dot{x}_2 &= \tfrac{1}{C}x_1 - \tfrac{1}{RC}x_2 - \tfrac{1}{C}x_1 u \\ y &= x_2 \end{aligned} \tag{23}$$

where $x_1 = I$, $x_2 = V$, and the coefficients are defined as $R = 30[\Omega]$, $C = 20[\mu\text{F}]$, $L = 20[\text{mH}]$, $E(w) = 15 + w[\text{V}]$, and $w = \sin(10t)$. Here, $\Delta E (= 0.5w^2 + 3)$ is unknown voltage source structure. The control objective is to obtain regulated $V = 30[\text{V}]$. The nominal equilibrium point is $\mathbf{x}_0(w) = [\frac{r_d{}^2}{E(w)R}, r_d]^T, \mathbf{u}_0 = 1 - \frac{E(w)}{r_d}$. Simulation is performed under $\Delta f_1 = \frac{\Delta E(w)}{L}$, $\underline{\alpha} = \frac{3}{L}$, and $\overline{\alpha} = \frac{3.5}{L}$. Fig 4. shows the output trajectory when only static state feedback controller($u = k(0,x)$) is employed and Fig 5. when the dynamic state feedback controller($u = k(z,x), \dot{z} = l(z,y)$) is employed. In Fig. 4-(d) shows some output error due to the nonzero $d(t), t \geq 0$. Fig. 5-(c) shows that $\tilde{\mathbf{u}}'_k$ has tendency to approach into $\tilde{\mathbf{u}}$ and Fig. 5-(d) shows well regulated output voltage.

3. CONCLUSIONS

In this paper, we proposed a dynamic state feedback controller which enables the state variables to asymptotically track the estimated equilibrium point obtained by NR-estimator. The overall system shows robustness for the uncertainties. The NR-estimator makes fast computation of equilibrium point and it can be applied to existing controllers which utilize equilibrium point for given system. In addition, more precise bound about uncertainty provides better regulation performance.

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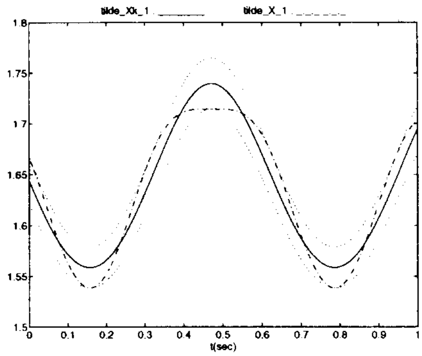

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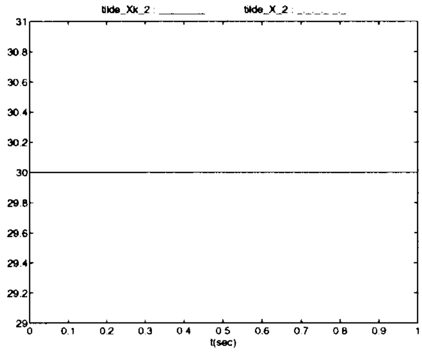

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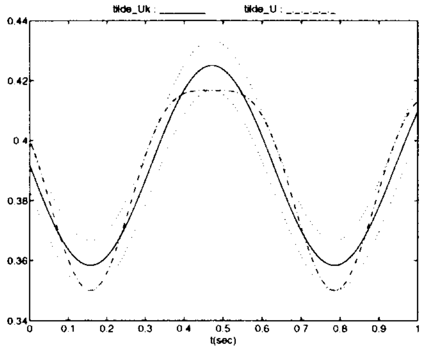

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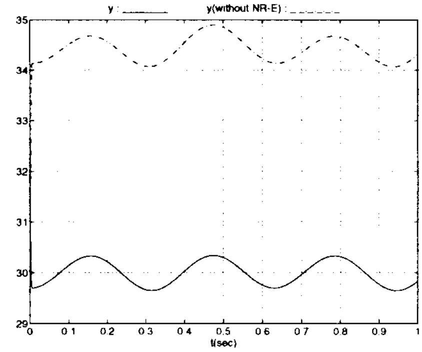
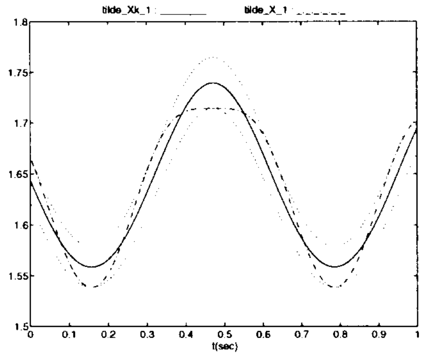


(d)

Fig. 4: Output when with no compensator : $u = k(0,x)$



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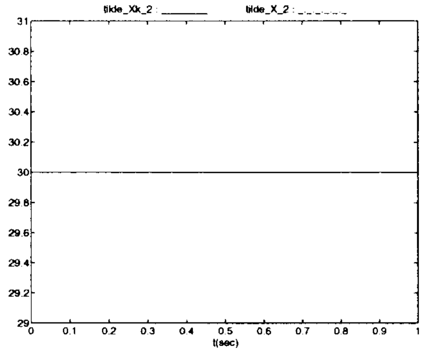

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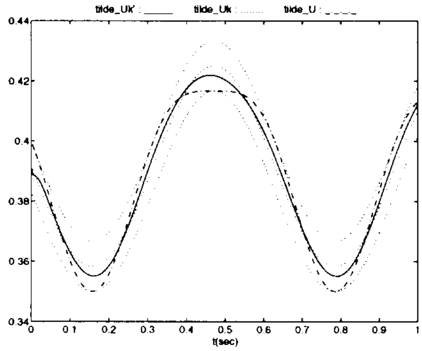

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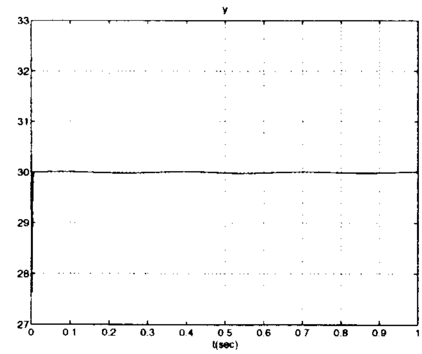

(d)

Fig. 5: Output when with compensator : $u = k(z,x), \dot{z} = l(z,y)$