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# Local Minimization Behavior of Weighted Kinematically Decoupled Joint Space Decomposition for Redundant Manipulators

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**Abstracts** Kinematically redundant manipulators have been studied because of its usefulness of kinematic redundancy. It is natural that the kinematic redundancy induces a kind of control redundancy. By using the weighted kinematically decoupled joint space decomposition, we unify the control redundancy and the kinematic redundancy parametrized by the joint space weighting matrix. Concentrating to the particular component of each decomposition, we can describe the local minimization behavior of the control weighted quadratic by each weighted decomposition. The result extends the conventional results on general setting, and should be of interest in understanding the motion behavior of kinematically redundant manipulators.

## 1. Introduction

Since the initial stage of studying redundant manipulators, the control, or torque, minimization capability has been under intensive researches, some of which are found in [2, 7, 5, 4, 1]. To proceed the discussion, we assume the followings. Given a redundant manipulators whose degrees of freedom is $n$ and task space dimension is $m(< n)$, the task motion $\boldsymbol{p} = (p_1, \cdots, p_m)^T$ is mapped from the joint motion $\boldsymbol{q} = (q_1, \cdots, q_n)^T$ by the set of forward kinematics

$$\begin{aligned} \boldsymbol{p} &= \boldsymbol{k}(\boldsymbol{q}) && (1)\\ \dot{\boldsymbol{p}} &= \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}}, \quad \boldsymbol{J}(\boldsymbol{q}) = \frac{\partial \boldsymbol{k}}{\partial \boldsymbol{q}^T} && (2)\\ \ddot{\boldsymbol{p}} &= \boldsymbol{J}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \dot{\boldsymbol{J}}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}}, && (3) \end{aligned}$$

where the $m \times n$-matrix $\boldsymbol{J}(\boldsymbol{q})$ is called the Jacobian matrix. Assume that by denoting the control by $n$-vector $\boldsymbol{\tau}$, the dynamics in the joint space is formulated by

$$\boldsymbol{\tau} = \boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{h}(\boldsymbol{q}, \dot{\boldsymbol{q}}), \quad (4)$$

where $\boldsymbol{M}(\boldsymbol{q}) \in \Re^{n\times n}$ is the inertia matrix, and $\boldsymbol{h}(\boldsymbol{q}, \dot{\boldsymbol{q}})$ denotes collectively all the nonlinear dynamics except the inertial torque. In the sequel, all the arguments are suppressed since no ambiguity would arise.

The primary objective of a manipulator is to track a desired task trajectory $\boldsymbol{p}_d(t)$ by manipulating the joint torque $\boldsymbol{\tau}$, referred to as the control. When the dynamic effect is negligible, which is true especially for the manipulator with large transmission reduction, which is operating slowly, the manipulator can be approximated by a simple linear system,

$$\dot{\boldsymbol{q}} = \boldsymbol{u}$$

by suitable definition of $\boldsymbol{u}$. Taking into account the output equation given by (1), the control problem reduces to the inverse kinematic system. It is well known that a possible joint velocity input $\boldsymbol{u}$ which reproduces the desired task trajectory can be obtained by

$$\dot{\boldsymbol{q}} = \dot{\boldsymbol{q}}_p + \dot{\boldsymbol{q}}_h = \boldsymbol{J}^{W+}(\dot{\boldsymbol{p}}_d + \boldsymbol{K}\boldsymbol{e}_p) + (\boldsymbol{I} - \boldsymbol{J}^{W+}\boldsymbol{J})\boldsymbol{\alpha} \quad (5)$$

where $\boldsymbol{J}^{W+} \in \Re^{n\times m}$ denotes the weighted pseudoinverse of the Jacobian $\boldsymbol{J}$, and $\boldsymbol{\alpha} \in \Re^n$ is arbitrary velocity. Note that the particular velocity $\dot{\boldsymbol{q}}_p$ is the unique joint velocity to reproduce the task velocity $\dot{\boldsymbol{p}}_d$ while minimizing the weighted velocity quadratic $\dot{\boldsymbol{q}}^T\boldsymbol{W}\dot{\boldsymbol{q}}$, and the homogeneous velocity $\dot{\boldsymbol{q}}_h$ can be used to impose self-motion. It should be noted that the homogeneous velocity, if any, increses the joint velocity quadratic $\dot{\boldsymbol{q}}^T\boldsymbol{W}\dot{\boldsymbol{q}}$ locally, but it can enforce the manipulator to move towards a desired configuration, which may result in a globally smaller velocity trajectory.

Discarding the assumptions leading to the linear system, the inverse dynamic control problem follows which solves $\boldsymbol{\tau}$ such that the manipulator can reproduce $\dot{\boldsymbol{p}}_d$. To find a control $\boldsymbol{\tau}$ minimizing $\boldsymbol{\tau}^T\boldsymbol{W}\boldsymbol{\tau}$ subject to (3) is the problem definition for the local torque minimization that is

> To find a inverse dynamic torque $\boldsymbol{\tau}$ at the state $(\boldsymbol{q}^T, \dot{\boldsymbol{q}}^T)^T$
> which minimizes
> $$\boldsymbol{\tau}^T\boldsymbol{W}\boldsymbol{\tau}$$
> subject to
> $$\ddot{\boldsymbol{p}}_{ref} = \boldsymbol{J}\ddot{\boldsymbol{q}} + \dot{\boldsymbol{J}}\dot{\boldsymbol{q}}.$$

Note that the reference acceleration $\ddot{\boldsymbol{p}}_{ref}$ is used instead of $\ddot{\boldsymbol{p}}_d$ or $\ddot{\boldsymbol{p}}$ to take into account the control due to possible task motion error. One of the conventional solutions follows the following calculations. Using the Lagrange multiplier technique the problem is converted to an unconstrained problem which finds a minimal solution of

$$L = \boldsymbol{\tau}^T\boldsymbol{W}\boldsymbol{\tau} + \boldsymbol{\lambda}^T(\boldsymbol{J}\ddot{\boldsymbol{q}} + \dot{\boldsymbol{J}}\dot{\boldsymbol{q}} - \ddot{\boldsymbol{p}}_{ref}).$$

The minimal control $\boldsymbol{\tau}$ is then obtained by solving

$$\frac{\partial L}{\partial \boldsymbol{\tau}^T} = 2\boldsymbol{\tau}^T \boldsymbol{W} - \boldsymbol{\Lambda}^T \boldsymbol{J}\boldsymbol{M}^{-1} = \boldsymbol{0}$$

from which

$$\begin{aligned} \boldsymbol{\tau} &= \boldsymbol{W}^{-1}\boldsymbol{M}^{-1}\boldsymbol{J}^T(\boldsymbol{J}\boldsymbol{M}^{-1}\boldsymbol{W}^{-1}\boldsymbol{M}^{-1}\boldsymbol{J}^T)^{-1} \\ &\cdot(\ddot{\boldsymbol{p}}_{ref} - \dot{\boldsymbol{J}}\dot{\boldsymbol{q}} + \boldsymbol{J}\boldsymbol{M}^{-1}\boldsymbol{h}). \end{aligned} \tag{6}$$

From the above discussion, we can see that there exist a unique control quadratic which is minimized locally according to the joint space weight matrix $\boldsymbol{W}$, that is which weighted pseudouinverse is used. In this article, we study this aspect. The kinematic redundancy appearing in (5) is generalized to inverse dynamic case using the notion of control redundancy. Then the local minimization behavior of control quadratic imposed by the weighted kinematically decoupled joint space decomposition [6] is described.

## 2. Parametrization and Representation of Redundancy

### 1. *Control redundancy*

By substituting $\ddot{\boldsymbol{q}}$ from (4) to (3), a kind of direct map from the control input to the task motion is obtained of the form

$$\boldsymbol{J}\boldsymbol{M}^{-1}\boldsymbol{\tau} = \ddot{\boldsymbol{p}} - \dot{\boldsymbol{J}}\dot{\boldsymbol{q}} + \boldsymbol{J}\boldsymbol{M}^{-1}\boldsymbol{h}. \tag{7}$$

This equation can be used to specify the portion of inverse dynamic torque, referred to as the *particular torque* $\boldsymbol{\tau}_p \in \Re^n$, to obtain the following closed-loop dynamics in the task space

$$\ddot{\boldsymbol{p}} = \ddot{\boldsymbol{p}}_{ref}. \tag{8}$$

By generating the particular torque such that

$$\boldsymbol{J}\boldsymbol{M}^{-1}\boldsymbol{\tau}_p = \ddot{\boldsymbol{p}}_{ref} - \dot{\boldsymbol{J}}\dot{\boldsymbol{q}} + \boldsymbol{J}\boldsymbol{M}^{-1}\boldsymbol{h} \tag{9}$$

the above task closed-loop dynamics is guaranteed. To see the effect of the kinematic redundancy on the inverse dynamic control, observe that the set of controls whose particular torques satisfy (9) forms a $r$-dimensional linear affine space, where $r = n - m$ is called the degrees of redundancy. It is a direct consequence of the fact that the control having $\boldsymbol{\tau}_p$ as the particular torque component is uniquely decomposed by

$$\boldsymbol{\tau} = \boldsymbol{\tau}_p + \boldsymbol{\tau}_h$$

where $\boldsymbol{\tau}_h \in \Re^n$, called the *homogeneous torque*, is chosen among the null space of $\boldsymbol{J}\boldsymbol{M}^{-1}$, whose dimension is $r$. Then $\boldsymbol{\tau}_p$ forms a $r$-dimenisonal vector subspace. The *control redundancy* describes the effect of the kinematic redundancy on the inverse dynamic control as described above.

It is definitely related to the conventional redundancy formulation in the velocity level, referred to as the kinematic redundancy. By integrating the task closed-loop dynamics (8), the actual task velocity will be $\dot{\boldsymbol{p}}_{ref}(t) = \int_{t_0}^{t} \ddot{\boldsymbol{p}}_{ref} + \dot{\boldsymbol{p}}(t_0)$. Then the joint velocity should have the *particular velocity* satisfying

$$\boldsymbol{J}\dot{\boldsymbol{q}}_p = \dot{\boldsymbol{p}}_{ref}. \tag{10}$$

Similarly, the set of $\dot{\boldsymbol{q}}$ having $\dot{\boldsymbol{q}}_p$ as the particular velocity forms a $r$-dimensional linear affine space. By decomposing $\dot{\boldsymbol{q}}$ by

$$\dot{\boldsymbol{q}} = \dot{\boldsymbol{q}}_p + \dot{\boldsymbol{q}}_h$$

where $\dot{\boldsymbol{q}}_h$ forms a $r$-dimenisonal vector subspace corresponding to the null space of $\boldsymbol{J}$.

It should be noted that the $\dot{\boldsymbol{q}}_h$ and $\boldsymbol{\tau}_h$ is not independent, as $\dot{\boldsymbol{q}}_p$ and $\boldsymbol{\tau}_p$ are related as above. To express the kinematic and control redundancy, a conventional approach is to decompose $\dot{\boldsymbol{q}}$ by

$$\dot{\boldsymbol{q}} = \boldsymbol{J}^{W+}\dot{\boldsymbol{p}}_{ref} + (\boldsymbol{I} - \boldsymbol{J}^{W+}\boldsymbol{J})\dot{\boldsymbol{q}}_h \tag{11}$$

where $\boldsymbol{J}^{W+} \in \Re^{n\times m}$ is the weighted pseudoinverse of $\boldsymbol{J}$. It is easy to see that the filter $(\boldsymbol{I} - \boldsymbol{J}^{W+}\boldsymbol{J})$ passes any homogeneous velocity without any modification, or

$$\dot{\boldsymbol{q}}_h = (\boldsymbol{I} - \boldsymbol{J}^{W+}\boldsymbol{J})\dot{\boldsymbol{q}}_h$$

for all $\dot{\boldsymbol{q}}_h \in \mathcal{N}(\boldsymbol{J})$.

### 2. *Kinematics decomposition*

It was shown [6] that the above decomposition is equivalent to

$$\dot{\boldsymbol{q}} = \boldsymbol{W}^{-\frac{1}{2}}\left[\boldsymbol{R}_{W^{-\frac{1}{2}}}(\boldsymbol{J}_{W^{-\frac{1}{2}}}\boldsymbol{R}_{W^{-\frac{1}{2}}})^{-1}\right.$$
$$\left|\boldsymbol{N}_{W^{-\frac{1}{2}}}(\boldsymbol{Z}_{W^{-\frac{1}{2}}}\boldsymbol{N}_{W^{-\frac{1}{2}}})^{-1}\right]\begin{pmatrix}\dot{\boldsymbol{p}}\\ \dot{\boldsymbol{n}}_{W^{\frac{1}{2}}}\end{pmatrix}, \tag{12}$$

where

$$\boldsymbol{J}_{W^{-\frac{1}{2}}} = \boldsymbol{J}\boldsymbol{W}^{-\frac{1}{2}}, \qquad \boldsymbol{Z}_{W^{-\frac{1}{2}}} = \boldsymbol{Z}\boldsymbol{W}^{\frac{1}{2}} \tag{13}$$

and $\boldsymbol{R}_{W^{-\frac{1}{2}}} \in \Re^{n\times m}$ and $\boldsymbol{N}_{W^{-\frac{1}{2}}} \in \Re^{n\times r}$ are numerically obtained by taking the eigenvalue decomposition of $\boldsymbol{J}^T_{W^{-\frac{1}{2}}}\boldsymbol{J}_{W^{-\frac{1}{2}}}$, i.e.

$$\boldsymbol{J}^T_{W^{-\frac{1}{2}}}\boldsymbol{J}_{W^{-\frac{1}{2}}} = \begin{bmatrix}\boldsymbol{R}_{W^{-\frac{1}{2}}} & \boldsymbol{N}_{W^{-\frac{1}{2}}}\end{bmatrix}\begin{bmatrix}\boldsymbol{\Sigma}^2_{W^{-\frac{1}{2}}} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{0}\end{bmatrix}$$
$$\begin{bmatrix}\boldsymbol{R}_{W^{-\frac{1}{2}}} & \boldsymbol{N}_{W^{-\frac{1}{2}}}\end{bmatrix}^T, \tag{14}$$

where $\boldsymbol{\Sigma}^2_{W^{-\frac{1}{2}}}$ is the diagonal matrix having $m$ nonzero eigenvalues. The $r$-dimensional vector $\dot{\boldsymbol{n}}_{W^{\frac{1}{2}}}$ is called the *(weighted) null velocity* defined by

$$\dot{\boldsymbol{n}} = \boldsymbol{Z}\boldsymbol{W}\dot{\boldsymbol{q}}. \tag{15}$$

The null velocity $\dot{n}_{W^{\frac{1}{2}}} \in \Re^r$ is a kind of minimal parametrization of $\dot{q}_h$ since $ZW$ passes only the homogeneous velocity because

$$\begin{aligned} ZW\dot{q} &= ZW(J^{W+}\dot{p}_{ref} + (I - J^{W+}J)\dot{q}_h) \\ &= ZWW^{-\frac{1}{2}}J^{+}_{W^{-\frac{1}{2}}}\dot{p}_{ref} \\ &\quad + ZWW^{-\frac{1}{2}}Z^{+}_{W^{-\frac{1}{2}}}Z_{W^{-\frac{1}{2}}}W^{\frac{1}{2}}\dot{q}_h \\ &= Z_{W^{-\frac{1}{2}}}J^{+}_{W^{-\frac{1}{2}}}\dot{p}_{ref} \\ &\quad + Z_{W^{-\frac{1}{2}}}Z^{+}_{W^{-\frac{1}{2}}}Z_{W^{-\frac{1}{2}}}W^{\frac{1}{2}}\dot{q}_h \\ &= Z_{W^{-\frac{1}{2}}}W^{\frac{1}{2}}\dot{q}_h \;=\; ZW\dot{q}_h. \end{aligned}$$

Similarly, by viewing the task velocity $\dot{p}$ as a minimal parametrization of $\dot{q}_p$ deined by (2), it forms a new coordinate together with $\dot{n}_{W^{\frac{1}{2}}}$ defined by

$$\begin{pmatrix} \dot{p} \\ \dot{n}_{W^{\frac{1}{2}}} \end{pmatrix} = \begin{bmatrix} J \\ ZW \end{bmatrix} \dot{q} \qquad (16)$$

whose inverse is given by (12). Note that the coordinate transformation is not singular if $J$ has full row rank $m$, i.e. kinematically nonsingular.

### *3. Dynamics decomposition*

Since time-derivative of (12) reduces to

$$\begin{aligned} \ddot{q} &= W^{-\frac{1}{2}}\Big[R_{W^{-\frac{1}{2}}}(J_{W^{-\frac{1}{2}}}R_{W^{-\frac{1}{2}}})^{-1} \\ &\quad \Big| N_{W^{-\frac{1}{2}}}(Z_{W^{-\frac{1}{2}}}N_{W^{-\frac{1}{2}}})^{-1}\Big] \\ &\quad \cdot \begin{pmatrix} \ddot{p} - \dot{J}\dot{q} \\ \ddot{n}_{W^{\frac{1}{2}}} - (\dot{Z}W + Z\dot{W})\dot{q} \end{pmatrix}, \end{aligned} \qquad (17)$$

the joint dynamics (4) can be expressed by [6]

$$\begin{aligned} \tau &= W^{-\frac{1}{2}}\Big[R_{W^{-\frac{1}{2}}}(J_{W^{-\frac{1}{2}}}R_{W^{-\frac{1}{2}}})^{-1} \\ &\quad \Big| N_{W^{-\frac{1}{2}}}(Z_{W^{-\frac{1}{2}}}N_{W^{-\frac{1}{2}}})^{-1}\Big] \\ &\quad \cdot \begin{pmatrix} \ddot{p} - \dot{J}\dot{q} \\ \ddot{n}_{W^{\frac{1}{2}}} - (\dot{Z}W + Z\dot{W})\dot{q} \end{pmatrix} + h \end{aligned} \qquad (18)$$

which is called the *acceleration decomposed dynamics*. By further decomposing the nonlinear dynamics $h$, we get

$$\begin{aligned} \tau &= W^{-\frac{1}{2}}\Big[R_{W^{-\frac{1}{2}}}(J_{W^{-\frac{1}{2}}}R_{W^{-\frac{1}{2}}})^{-1} \\ &\quad \Big| N_{W^{-\frac{1}{2}}}(Z_{W^{-\frac{1}{2}}}N_{W^{-\frac{1}{2}}})^{-1}\Big] \\ &\quad \cdot \begin{pmatrix} \ddot{p} - \dot{J}\dot{q} + JM^{-1}h \\ \ddot{n}_{W^{\frac{1}{2}}} - (\dot{Z}W + Z\dot{W})\dot{q} + ZWM^{-1}h \end{pmatrix} \end{aligned} \qquad (19)$$

which is called the *weighted kinematically decomposed dynamics*. The first element of the right hand side vector is called the *task dynamic acceleration*, and the lower one is called the *null dynamic acceleration*.

Remark **2..1** *Since (17) is a proper inverse of (3), (18) should be regarded as an improved version of conventional acceleration decomposed dynamics*

$$\tau = M\left(J^{W+}(\ddot{p} - \dot{J}\dot{q}) + (I - J^{W+}J)\ddot{q}_h\right) + h. \quad (20)$$

*Also, (19) is also a generalized and improved version of conventional inverse of (7) since the latter formulated by*

$$\tau = (JM)^{+}(\ddot{p} - \dot{J}\dot{q} + JM^{-1}h) + (I - (JM)^{+}JM)\tau_h \quad (21)$$

*reduces to the case of the $M^2$-weighted kinematically decomposed dynamics. Note that two forms given in (20) and (21) are not equivalent to each other, whereas the proposed formulations of (18) and (19) are equivalent.*

## 3. Minimization Behaviors of Decomposed Inverse Dynamics

In this and subsequent sections, we will study on the behavioral characteristics which is imposed by a certain weight matrix. We proceed by introducing a concept of decoupling with respect to a weight matrix.

DEFINITION **3..1** *A quadratic form of joint velocity given by*

$$\dot{q}^T K \dot{q}, \quad K = K^T > 0$$

*is said to be decoupled with respect to $W$ if there exist symmetric positive definite weight matrices $W$, $K_p$, and $K_v$ of appropriate dimensions where the quadratic form can be transformed to*

$$\dot{q}^T K \dot{q} = \dot{p}^T K_p \dot{p} + \dot{n}^T_{W^{\frac{1}{2}}} K_n \dot{n}_{W^{\frac{1}{2}}} \qquad (22)$$

*with respect to the $W$-weighted kinematically decoupled joint space.*

The term 'decoupling' is obvious since in the new coordinate the quadratic does not include the coupled term betwwen $\dot{p}$ and $\dot{n}_{W^{\frac{1}{2}}}$. The matrix $W$ is called the decoupling matrix of the velocity quadratic. The decoupling matrix is not trivial. For example, let us transform the quadratic to the kinematically decoupled coordinate using $I$ as the weight. Then (12) being plugged in, there follows

$$\begin{aligned} \dot{q}^T K \dot{q} &= \dot{p}^T (JR)^{-T} R^T K R (JR)^{-1} \dot{p} \\ &+ \dot{n}^T (ZN)^{-T} N^T K N (ZN)^{-1} \dot{n} \\ &+ 2\,\dot{p}^T (JR)^{-T} R^T K N (ZN)^{-1} \dot{n}, \end{aligned}$$

which is not decoupled with respect to $I$ unless $K = I$. The following lemma establishes the obvious fact that a $W$-weighted velocity quadratic is decoupled with respect to the weight $W$.

LEMMA **3..1** *The quadratic form $\dot{q}^T W \dot{q}$ is decoupled with respect to the same weight matrix $W$. Moreover,*

$$\dot{q}^T W \dot{q} = \dot{p}^T (J W^{-1} J^T)^{-1} \dot{p} + \dot{n}^T_{W^{\frac{1}{2}}} (Z W Z^T)^{-1} \dot{n}_{W^{\frac{1}{2}}}. \quad (23)$$

*Proof.* The proof is not difficult since the direct substitution will verify the lemma. ∎

The concept of decoupling can be extended to the inverse dynamic control.

DEFINITION 3..2 *A $K$-weighted control quadratic given by*

$$\tau^T K \tau, \quad K = K^T > 0$$

*is said to be decoupled with respect to $W$ if there exist a symmetric positive-definite weight matrix $W$ by which the quadratic form can be transformed to one which does not contain a coupling term between the task dynamic acceleration and the null dynamic acceleration.*

The following lemma will answer the useful fact about the decoupling weight of a weighted control quadratic.

LEMMA 3..2 *The quadratic form $\tau^T K \tau$ is decoupled with respect to the weight matrix $W = MKM$. Moreover,*

$$\begin{aligned} \tau^T K \tau &= (\ddot{p} - \dot{J}\dot{q} + JM^{-1}h)^T (JW^{-1}J^T)^{-1} \\ &\quad \cdot(\ddot{p} - \dot{J}\dot{q} + JM^{-1}h) \qquad (24) \\ &+(\ddot{n}_{W^{\frac{1}{2}}} - (\dot{Z}W + Z\dot{W})\dot{q} + ZWM^{-1}h)^T (ZWZ^T)^{-1} \\ &\quad \cdot(\ddot{n}_{W^{\frac{1}{2}}} - (\dot{Z}W + Z\dot{W})\dot{q} + ZWM^{-1}h) \end{aligned}$$

The proof is direct and easy.

### *1. Local minimization of velocity quadratic*

Since a quadratic form $\dot{q}^T W \dot{q}$ is decoupled between $\dot{p}$ and $\dot{n}_{W^{\frac{1}{2}}}$, it is extremely useful to solve the following equality-constrained local minimization problem of the weighted quadratic

To find a $\dot{q}$ which minimizes

$$\dot{q}^T W(q) \dot{q}$$

subject to (2).

The problem can be reduced to an unconstrained quadratic minimization problem by parametrizing the solution using (12). The choice of the same weight matrix $W$ in weighted decomposition is guided by Lemma 3..1. Then the problem is recast as

To find a $\dot{n}_{W^{\frac{1}{2}}} \in \Re^r$ which minimizes

$$\begin{aligned} \dot{q}^T W \dot{q} &= \dot{p}^T \left(JW^{-1}J^T\right)^{-1} \dot{p} \\ &+ \dot{n}^T_{W^{\frac{1}{2}}} \left(ZWZ^T\right)^{-1} \dot{n}_{W^{\frac{1}{2}}}. \end{aligned}$$

Then the solution is easily obtained by requiring that

$$\frac{\partial \dot{q}^T W \dot{q}}{\partial \dot{n}^T_{W^{\frac{1}{2}}}} = \mathbf{0}. \qquad (25)$$

Evaluating the partial derivatives, we get

$$\left(ZWZ^T\right)^{-1} \dot{n}_{W^{\frac{1}{2}}} = \mathbf{0}.$$

Since the coeffient matrix is nonsingular, the solution is $\dot{n}_{W^{\frac{1}{2}}} = \mathbf{0}$. The optimal joint velocity is then given by

$$\begin{aligned} \dot{q} &= W^{-\frac{1}{2}} R_{W^{-\frac{1}{2}}} \left(J_{W^{-\frac{1}{2}}} R_{W^{-\frac{1}{2}}}\right)^{-1} \dot{p} \\ &= W^{-\frac{1}{2}} J^+_{W^{-\frac{1}{2}}} \dot{p} = J^{W+} \dot{p}, \qquad (26) \end{aligned}$$

which coincides with the well-known fact that the particular velocity by the $W$-weighted pseudoinverse minimizes the $W$-weighted velocity quadratic locally. The minimum quadratic is evaluated to be

$$\min \ \dot{q}^T W \dot{q} = \dot{p}^T (JW^{-1}J)^{-1} \dot{p}. \qquad (27)$$

Remark 3..1 *It is worth noting that the simple derivation of the result is owed to the weighted kinematically decoupled joint velocity decomposition with respect to the decoupling weight. For example, a different weight can be used to decompose the joint velocity instead of the decoupling matrix. For complexity comparison, assume that $I$ is used as the weight matrix. Then it is easy to see that the minimizing solution $\dot{n}$ is given by*

$$\dot{n} = -(ZN)(N^T W N)^{-1} N^T W R (JR)^{-1} \dot{p}$$

*which leads to the minimum quadratic value of*

$$\begin{aligned} \dot{q}^T W \dot{q} &= \dot{p}^T (JR)^{-T} R^T W (I - N(N^T W N)^{-1} N^T W) \\ &\quad \cdot R (JR)^{-1} \dot{p}. \end{aligned}$$

*Since the quadratic has a unique minimum, it should be equal to (27). Hence, the joint solutions should be equal. But the degrees of difficulty in two approaches are quite different. Thus we can conclude that to minimize a weighted quadratic the joint space decomposition using the decoupling matrix as the weight is most suitable.*

A class of weighted velocity quadratic includes various kinematic and dynamic related measures. Three important quadratics are the joint velocity norm, the kinetic energy, and the momentum norm, each of which can be defined as the $I$, $M$, and $M^2$-weighted velocity quadratic, respectively. The minimizing behavior of each quadratic is important in view that they inevitably affects the motion of a manipulator.

We can define three weighted kinematically decoupled joint space decompositions which decouples each quadratic form. The first is the kinematically decoupled joint space decomposition which decouples the velocity quadratic. The next is the $M$-weighted kinematically decoupled joint space decomposition which decouples the kinetic energy. The third is the squared-inertia weighted decomposition, with respect to which the momentum quadratic is decoupled.

Example 3..1 *(Velocity quadratic minimization)* The optimal velocity which minimizes $\dot{q}^T \dot{q}$ is

$$\dot{q} = R(JR)^{-1} \dot{p}. \qquad (28)$$

The associated minimum velocity quadratic is $\dot{p}^T (JJ^T)^{-1} \dot{p}$.

Example 3..2 *(Kinetic energy minimization)* The minimizing velocity of $\dot{q}^T M \dot{q}$

$$\dot{q} = M^{-\frac{1}{2}} R_{M^{-\frac{1}{2}}} \left( J_{M^{-\frac{1}{2}}} R_{M^{-\frac{1}{2}}} \right)^{-1} \dot{p}, \qquad (29)$$

with the minimum kinetic energy $\dot{p}^T (J M^{-1} J^T)^{-1} \dot{p}$.

Example 3..3 *(Momentum quadratic minimization)* The minimization of the momentum quadratic $\dot{q}^T M^2 \dot{q}$ leads to

$$\dot{q} = M^{-1} R_{M^{-1}} \left( J_{M^{-1}} R_{M^{-1}} \right)^{-1} \dot{p}. \qquad (30)$$

and the minimum momentum is $\dot{p}^T (J M^{-2} J^T)^{-1} \dot{p}$.

*2. Local minimization of control quadratic*

In this section, the local minimizing solution of a weighted control quadratic $\tau^T K \tau$ is found. This result is specialized to cases of $I$, $M^{-1}$, and $M^{-2}$-weighted control quadratic. As a consequence, the relation between local minimization of the weighted velocity quadratic and control quadratic is revealed.

Let us formulate the general solution of the following problem

To find a solution $\tau$ which minimizes

$$\tau^T K(q) \tau$$

subject to (7).

Remark 3..2 *The equality constraint (7) is equivalent to (3) under the joint dynamics (4).*

Note that the quadratic is decoupled with respect to $W = MKM$ and the quadratic is transformed to (24). Since the only variable that can be manipulated is $\ddot{n}_{W^{\frac{1}{2}}}$, the problem is reduced to the unconstrained minimization problem which finds $\ddot{n}_{W^{\frac{1}{2}}}$ minimizing the transformed quadratic.

Then by solving

$$\frac{\partial \tau^T K \tau}{\partial \ddot{n}^T_{W^{\frac{1}{2}}}} = 0$$

we obtain

$$\ddot{n}^*_{W^{\frac{1}{2}}} = (\dot{Z} W + Z \dot{W}) \dot{q} - Z M K h, \qquad (31)$$

where $W = MKM$. The solution can be considered as the minimal null dynamic acceleration

$$0 = \ddot{n}^*_{W^{\frac{1}{2}}} - (\dot{Z} W + Z \dot{W}) \dot{q} + Z M K h. \qquad (32)$$

With this optimal solution, the minimizing control is given by

$$\tau = M W^{-\frac{1}{2}} R_{W^{-\frac{1}{2}}} \left( J_{W^{-\frac{1}{2}}} R_{W^{-\frac{1}{2}}} \right)^{-1} \cdot (\ddot{p} - \dot{J} \dot{q} + J M^{-1} h). \qquad (33)$$

Also, the minimal quadratic is

$$\tau^T K \tau = \left\{ \ddot{p} - \dot{J} \dot{q} + J M^{-1} h \right\}^T (J W^{-1} J^T)^{-1} \cdot \left\{ \ddot{p} - \dot{J} \dot{q} + J M^{-1} h \right\}. \qquad (34)$$

Using the result, three speical cases of weighted quadratic are analyzed, where $K$ is $I$, $M^{-1}$, and $M^{-2}$.

Example 3..4 *(Torque quadratic minimization)* The optimal control which minimizes $\tau^T \tau$ is obtained by the $M^2$-weighted kinematically decomposed dynamics by setting the null dynamic acceleration zero, i.e.

$$\tau = R_{M^{-1}} \left( J_{M^{-1}} R_{M^{-1}} \right)^{-1} (\ddot{p} - \dot{J} \dot{q} + J M^{-1} h). \qquad (35)$$

Example 3..5 *(Inertia-inverse weighted torque quadratic minimization)* To minimize $\tau^T M^{-1} \tau$, by setting $K = M^{-1}$ and $W = M$ the minimizing control is

$$\tau = M^{\frac{1}{2}} R_{M^{-\frac{1}{2}}} \left( J_{M^{-\frac{1}{2}}} R_{W^{-\frac{1}{2}}} \right)^{-1} (\ddot{p} - \dot{J} \dot{q} + J M^{-1} h). \qquad (36)$$

Example 3..6 *(Squared inertia-inverse weighted torque quadratic Minimization)* The torque given by

$$\tau = M R (J R)^{-1} \left\{ \ddot{p} - \dot{J} \dot{q} J M^{-1} h \right\} \qquad (37)$$

locally minimizes $\tau^T M^{-2} \tau$.

Note that it is equal to the pseudoinverse solution of $J M^{-1}$ of the following task space open-loop dynamics

$$J M^{-1} \tau = \ddot{p} - \dot{J} \dot{q} + J M^{-1} h. \qquad (38)$$

By observing that the weight matrix is $M^2$, we can conclude that the squared inertia weighted kinematically decoupled joint space decomposition has the property of partial minimization of $\dot{q}^T M^2 \dot{q}$ and $\tau^T \tau$. Note that the above solution is equal to the $M^{-1}$-weighted pseudoinverse solution of (38). In this case, we can conclude that the inertia weighted kinematically decoupled joint space decomposition impose the local behavior of partial minimization of $\dot{q}^T M \dot{q}$ and $\tau^T M^{-1} \tau$. By taking the $M^{-2}$-weighted pseudoinverse of $J M^{-1}$ for (38), we can get the same solution. Hence, the kinematically decoupled joint space decomposition has the local behavior partially minimizing $\dot{q}^T \dot{q}$ and $\tau^T M^{-2} \tau$.

*3. Relation of local minimization behavior*

Observing the solutions to local minimization of velocity quadratic and control quadratic reveals that the optimal velocity minimizing $\dot{q}^T W \dot{q}$ and $\tau^T M^{-1} W M^{-1} \tau$ are particular component using the $J^{W+}$. In other words, they are particular components of the inverse equation to (2) and (7) with respect to $W$-weighted kinematically decoupled joint space decomposition. Focusing on the three decompositions discussed in previous two sections, they are summarized in Tables 1 and 2.

Remark 3..3 *It can be shown [5, 3] that the locally minimizing solution of the weighted torque quadratic $\boldsymbol{\tau}^T \boldsymbol{M}^{-1}\boldsymbol{W}\boldsymbol{M}^{-1}\boldsymbol{\tau}$ corresponds to the globally minimizing solution of the weighted velocity quadratic $\dot{\boldsymbol{q}}^T\boldsymbol{W}\dot{\boldsymbol{q}}$ given by*

$$\int_{t_0}^{t_f} \dot{\boldsymbol{q}}^T \boldsymbol{W}\dot{\boldsymbol{q}}dt. \tag{39}$$

It should be noted that the optimal solution to local control quadratic is fundamentally different from one to local or global velocity quadratic in the following sense. In case of velocity minimization, the optimal solution leads to zero null velocity during motion, since

$$\boldsymbol{ZWW}^{-\frac{1}{2}}\boldsymbol{R}_{W^{-\frac{1}{2}}}\left(\boldsymbol{J}_{W^{-\frac{1}{2}}}\boldsymbol{R}_{W^{-\frac{1}{2}}}\right)^{-1}\dot{\boldsymbol{p}} = \boldsymbol{0}.$$

Unfortunately, the resulting null dynamics from the minimal control is not trivial, but of the form

$$\ddot{\boldsymbol{n}}_{W^{\frac{1}{2}}} - (\dot{\boldsymbol{Z}}\boldsymbol{W} + \boldsymbol{Z}\dot{\boldsymbol{W}})\dot{\boldsymbol{q}} + \boldsymbol{ZWM}^{-1}\boldsymbol{h} = \boldsymbol{0}.$$

Decomposing $\dot{\boldsymbol{q}}$ yields

$$\ddot{\boldsymbol{n}}_{W^{\frac{1}{2}}} - (\dot{\boldsymbol{Z}}\boldsymbol{W}+\boldsymbol{Z}\dot{\boldsymbol{W}})\boldsymbol{W}^{-\frac{1}{2}}\boldsymbol{N}_{W^{-\frac{1}{2}}}\left(\boldsymbol{Z}_{W^{-\frac{1}{2}}}\boldsymbol{N}_{W^{-\frac{1}{2}}}\right)^{-1}\dot{\boldsymbol{n}}_{W^{\frac{1}{2}}}$$
$$= (\dot{\boldsymbol{Z}}\boldsymbol{W} + \boldsymbol{Z}\dot{\boldsymbol{W}})\boldsymbol{W}^{-\frac{1}{2}}\boldsymbol{R}_{W^{-\frac{1}{2}}}\left(\boldsymbol{J}_{W^{-\frac{1}{2}}}\boldsymbol{R}_{W^{-\frac{1}{2}}}\right)^{-1}\dot{\boldsymbol{p}} - \boldsymbol{ZWM}^{-1}\boldsymbol{h}$$

. In other words, the null motion is internally generated by the above nontrivial dynamics.

## 4. Conclusions

In this article, we have described the minimization behavior according to which weighted pseudoinverse is used to invert the kinematics based on the weighted kinematically decoupled joint space decomposition. Thanks to the minimal and decoupled expression of the kinematic and control redundancy parametrized by each weight matrix, the problems of local minimization of velocity and control quadratic reduce to rather trivial ones. The work of this article will be useful in designing an inverse dynamic controller to locally minimizes a certain control quadratic.

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Table 1: Velocity quadratic minimization

| | $\dot{\boldsymbol{p}} = \boldsymbol{J}\dot{\boldsymbol{q}}$ | minimizing norm |
|---|---|---|
| $\boldsymbol{J}^+$ | $\dot{\boldsymbol{q}} = \boldsymbol{R}(\boldsymbol{JR})^{-1}\dot{\boldsymbol{p}}$ | $\dot{\boldsymbol{q}}^T\dot{\boldsymbol{q}}$ |
| $\boldsymbol{J}^{M+}$ | $\dot{\boldsymbol{q}} = \boldsymbol{M}^{-\frac{1}{2}}\boldsymbol{R}_{M^{-\frac{1}{2}}}\left(\boldsymbol{J}_{M^{-\frac{1}{2}}}\boldsymbol{R}_{M^{-\frac{1}{2}}}\right)^{-1}\dot{\boldsymbol{p}}$ | $\dot{\boldsymbol{q}}^T\boldsymbol{M}\dot{\boldsymbol{q}}$ |
| $\boldsymbol{J}^{M^2+}$ | $\dot{\boldsymbol{q}} = \boldsymbol{M}^{-1}\boldsymbol{R}_{M^{-1}}(\boldsymbol{J}_{M^{-1}}\boldsymbol{R}_{M^{-1}})^{-1}\dot{\boldsymbol{p}}$ | $\dot{\boldsymbol{q}}^T\boldsymbol{M}^2\dot{\boldsymbol{q}}$ |

Table 2: Joint Torque Minimization

| | $\boldsymbol{JM}^{-1}\boldsymbol{\tau} = \ddot{\boldsymbol{p}} - \dot{\boldsymbol{J}}\dot{\boldsymbol{q}} + \boldsymbol{JM}^{-1}\boldsymbol{h}$ | minimizing norm |
|---|---|---|
| $\boldsymbol{J}_{M^{-1}}^{M^{-2}+} = \boldsymbol{MJ}^+$ | $\boldsymbol{\tau} = \boldsymbol{MR}(\boldsymbol{JR})^{-1}\left\{\ddot{\boldsymbol{p}} - \dot{\boldsymbol{J}}\dot{\boldsymbol{q}} + \boldsymbol{JM}^{-1}\boldsymbol{h}\right\}$ | $\boldsymbol{\tau}^T\boldsymbol{M}^{-2}\boldsymbol{\tau}$ |
| $\boldsymbol{J}_{M^{-1}}^{M^{-1}+} = \boldsymbol{MJ}^{M+}$ | $\boldsymbol{\tau} = \boldsymbol{M}^{\frac{1}{2}}\boldsymbol{R}_{M^{-\frac{1}{2}}}\left(\boldsymbol{J}_{M^{-\frac{1}{2}}}\boldsymbol{R}_{M^{-\frac{1}{2}}}\right)^{-1}\left\{\ddot{\boldsymbol{p}} - \dot{\boldsymbol{J}}\dot{\boldsymbol{q}} + \boldsymbol{JM}^{-1}\boldsymbol{h}\right\}$ | $\boldsymbol{\tau}^T\boldsymbol{M}^{-1}\boldsymbol{\tau}$ |
| $\boldsymbol{J}_{M^{-1}}^{+} = \boldsymbol{MJ}^{M^2+}$ | $\boldsymbol{\tau} = \boldsymbol{R}_{M^{-1}}(\boldsymbol{J}_{M^{-1}}\boldsymbol{R}_{M^{-1}})^{-1}\left\{\ddot{\boldsymbol{p}} - \dot{\boldsymbol{J}}\dot{\boldsymbol{q}} + \boldsymbol{JM}^{-1}\boldsymbol{h}\right\}$ | $\boldsymbol{\tau}^T\boldsymbol{\tau}$ |