Proceedings of the 11th
KACC, October 1996

DEVELOPMENTS IN ROBUST STOCHASTIC CONTROL: RISK-SENSITIVE AND MINIMAL COST VARIANCE CONTROL

Chang-Hee Won
TT & C Section, ETRI
Yusong P.O. Box 106
Taejon, Korea 305-600
Tel: (82-42) 860-4876; Fax: (82-42) 860-6949; Email: cwon@kepler.etri.re.kr

Abstract

Continuing advances in the formulation and solution of risk-sensitive control problems have reached a point at which this topic is becoming one of the more intriguing modern paradigms of feedback thought. Despite a prevailing atmosphere of close scrutiny of theoretical studies, the risk-sensitive body of knowledge is growing. Moreover, from the point of view of applications, the detailed properties of risk-sensitive design are only now beginning to be worked out. Accordingly, the time seems to be right for a survey of the historical underpinnings of the subject. This paper addresses the beginnings and the evolution, over the first quarter-century or so, and points out the close relationship of the topic with the notion of optimal cost cumulants, in particular the cost variance. It is to be expected that, in due course, some duality will appear between these notions and those in estimation and filtering. The purpose of this document is to help to lay a framework for that eventuality.

Keywords Stochastic, Robust, Risk-Sensitive, Minimal Cost Variance

1. Introduction

For a number of years, researchers have been pointing out that the time domain characterization of H_∞ controllers contains a "generalized" Riccati type equation that is also found in linear-quadratic zero-sum differential games and in risk sensitive linear-exponential-quadratic stochastic control. See, for example, [11]. This seems to suggest the possibility of a "grand synthesis" of different approaches to robust control. See Figure 1 for an overview of some of the connections among these various different areas of robust control.

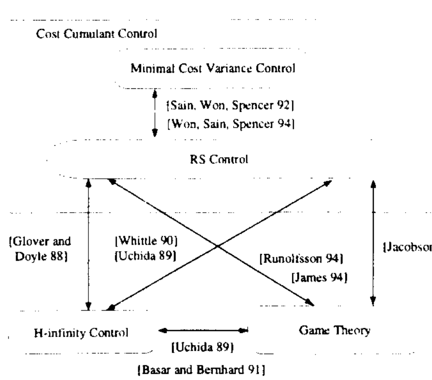

Figure 1: Relations Between Various Robust Controls

The beginning of stochastic control may go all the way back to 1940s when N. Wiener published his results on mean square filtering for weapons fire control developed during World War II. Since then stochastic control went through much theoretical advances by the renowned researchers such as Kalman, Bucy, Wonham, Kushner, Athans, to just name a few [16]. During this time linear quadratic regulator (LQR), linear quadratic Gaussian (LQG) problems have been solved in both full state and output feedback cases. For LQG results refer to excellent texts such as [5, 7, 8]. By the time Doyle and Stein introduced loop transfer recovery (LTR) method in the seventies, stochastic control seemed to have reached its peak. It is around this time that Jacobson came up with the risk-sensitive (RS) idea where the exponential of the quadratic cost function got optimized instead of just the queadratic cost function [13]. This RS idea developed further in the eighties by the researchers such as Speyer, Whittle, and Bensoussan [2, 28, 29, 32]. On the other hand, Minimal Cost Variance(MCV) control, where the idea is to minimize the variance (i.e., the second cumulant) of the cost function instead of the mean, has been developing from the sixties [23, 24, 25, 34]. Nowadays RS and MCV control are being actively investigated in the area of stochastic control [4, 10, 33, 34].

2. A Brief History of Risk-Sensitive Control

Risk-sensitive optimal control seems to have started with Jacobson in 1973. In the 1970s Jacobson extended LQG results by replacing the quadratic criterion with an exponential of a quadratic criterion, and related linear-exponential-quadratic-Gaussian (LEQG) control to differential games [13]. Many years later, Whittle noted Jacobson's results as a risk-sensitive control [32]. Speyer *et al.* [28] extended Jacobson's results to the noisy linear measurements case in discrete time. In [28], optimal control becomes a linear function of the smoothed history of the state, and the solutions are acquired by defining an enlarged state space composed of the entire state history. This enlarged state vector grows at every new stage but retains the feature of being a discrete linear system with additive white Gaussian noise. They also briefly discuss the continuous time terminal LEQG problem in [28], and the solutions are achieved by taking a formal limit of the discrete case solutions. In 1976, Speyer considered the noisy measurement case again in continuous time, but with zero state weighting in the cost function [29]. Unlike the previous work [28], the Hamilton-Jacobi-Bellman equation was used to produce the solutions in [29]. Kumar and van Schuppen derived the general solution of the partially observed exponential-of-integral (EOI) problem in continuous time with zero plant noise in 1981 [15]. In 1981, Whittle then published his results for the general solution of the partially observed LEOI problem in discrete time. Four years later, Bensoussan and van Schuppen reported the solution to the general case of a continuous time partially observed stochastic EOI problem using a different method from Whittle [2]. Unexpectedly in 1988, Glover and Doyle related H_∞ and minimum entropy criteria to the infinite horizon version of LEOI theory in discrete time, thus establishing a relationship between Whittle's risk-sensitive control and H_∞ optimal control [11]. This result was extended to continuous time by Glover [12]. In 1990, Whittle published the risk-sensitive maximum principle in book form [31], and published a journal article about the risk-sensitive maximum principle for the case of partially observed states using large deviation ideas [32]. A couple years later, Bensoussan published a book with all solutions (including the partially observed case) of the exponential-of-integral problem [3]. Başar and Bernhard noted the relationship between deterministic dynamic games and H_∞ optimal control in their book [1]. Also, the relationship between H_2 and H_∞ control has been studied in [6]. In 1992, James states that the risk-sensitive optimal control problem with full-state-feedback information is equivalent to a stochastic differential game problem. Fleming and McEneaney pointed out independent, but similar, results in 1992. In 1993, Won, Sain, and Spencer used Runolfsson's infinite horizon LEOI control results in a structural control application [33]. Hopkins presented discounted EOI solutions in 1994. More recently, in 1994, James, *et al.* published risk-sensitive control and dynamic games solutions for partially observed discrete-time nonlinear systems [14]. Runolfsson presented the relationship between Whittle's risk-sensitive control and stochastic differential games in the infinite-horizon case using large deviation ideas [21]. Most recently, Fleming and McEneaney published infinite-horizon full state feedback RS control results for nonlinear system and nonquadratic cost; and Bensoussan and Eliott published extensions of finite time partially observed RS contro! l results in *SIAM Journal on Control and Optimization*[10, 4].

3. A Brief History of Minimal Cost Variance Control

The fundamental idea behind minimal cost variance (MCV) control is to minimize the variance of the cost function while keeping the mean at a prespecified level. MCV control may be viewed in a broader context, which we shall call cost cumulant control. In cost cumulant control, certain linear combinations of the cost cumulants are constrained or minimized. Thus, the classical minimal mean cost problem can be seen as a special case of cost cumulant control, in which the first cumulant is minimized. This idea of minimizing the expected value of the cost function was developed in the 1960s; for example see [7, 30]. MCV control, which is also a form of cost cumulant control, was first developed in a dissertation in 1965 [22], and appeared in a journal in 1966 [23]. In 1968, Sain and Souza examined the minimal cost variance concept for problems of estimation[24]. In 1971, Sain and Liberty published an open loop result on minimizing the performance variance while keeping the performance mean at or below a prespecified value [25]. Liberty continued to study characteristic functions of integral quadratic forms, further developing the open loop MCV control idea in a Hilbert space setting [18]. Some years later, Liberty and Hartwig published the results of generating cost cumulants in the time domain [19]. Recently, Sain, Won, and Spencer showed that MCV control is an approximation of risk-sensitive control under some appropriate assumptions [26, 33]. Finally, feedback MCV control results are given in [34].

3.1. Open Loop Minimal Cost Variance Control

Consider a linear system

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + C(t)w(t) \tag{1}$$

and the performance measure

$$J = \int_0^{t_F} [x'(t)Qx(t) + u'(t)Ru(t)]\, dt + x'(t_F)Px(t_F) \tag{2}$$

where $w(t)$ is zero mean with white characteristics relative to the system, t_F is the fixed final time, $x(t) \in \mathbb{R}^n$ is the state of the system, and $u(t) \in \mathbb{R}^m$ is the control action. The weighting matrices P and Q are symmetric and positive semidefinite. And R is a symmetric and positive definite matrix. Note that

$$E\{w(t)w'(\sigma)\} = S\delta(t-\sigma) \tag{3}$$

where δ denotes the Dirac delta function and the ($'$) denotes the transposition.

The fundamental idea behind minimal cost variance control [22, 23] is to minimize the variance of the cost function J

$$J_{MV} = VAR_k\{J\} \tag{4}$$

while satisfying a constraint

$$E_k\{J\} = M \tag{5}$$

where J is the cost function and the subscript k on E denotes the expectation based upon a control law k generating the control action $u(t)$ from the state $x(t)$ or from a measurement history arising from that state. By means of a Lagrange multiplier μ, corresponding to the constraint (5), one can form the function

$$J_{MV} = \mu(E_k\{J\} - M) + VAR_k J, \tag{6}$$

which is equivalent to minimizing

$$\tilde{J}_{MV} = \mu E_k\{J\} + VAR_k\{J\}. \tag{7}$$

In [25], a Riccati solution to $\tilde{J}_{MV}$ minimization is developed for the open loop case

$$u(t) = k(t, x(0)). \tag{8}$$

The solution is based upon the differential equations

$$\dot{z}(t) = A(t)z(t) - \frac{1}{2}B(t)R^{-1}B'(t)\hat{\rho}(t) \tag{9}$$

$$\dot{\hat{\rho}}(t) = -A'(t)\hat{\rho}(t) - 2Qz(t) - 8\mu Qv(t) \tag{10}$$

$$\dot{v}(t) = A(t)v(t) + C(t)SC'(t)y(t) \tag{11}$$

$$\dot{y}(t) = -A'(t)y(t) - Qz(t) \tag{12}$$

with boundary conditions

$$z(0) = x(0) \quad (13)$$

$$\hat{\rho}(t_F) = 2Pz(t_F) + 8\mu Pv(t_F) \quad (14)$$

$$v(0) = 0 \quad (15)$$

$$y(t_F) = Pz(t_F) \quad (16)$$

and the control action relationship

$$u(t) = -\frac{1}{2}R^{-1}B'(t)\hat{\rho}(t). \quad (17)$$

3.2. Feedback Minimal Cost Variance Control

Consider the Ito sense stochastic diferential equation (SDE) with control,

$$dx(t) = f(t, x(t), u(t))dt + \sigma(t, x(t))dw(t). \quad (18)$$

And the cost function

$$J(t, x(t), k) = \int_t^{t_F} \Big[L(s, x(s), k(s, x(s)))\Big] ds + \psi(x(t_F)). \quad (19)$$

In MCV control we define a class of admissible controllers, then the cost variance is minimized within that class of controllers [27]. Define

$$V_1(t, x; k) = E\{J(t, x(t), k)|x(t) = x\}$$

$$V_2(t, x; k) = E\{J^2(t, x(t), k)|x(t) = x\}.$$

A minimal mean cost control law k_M^* satisfies $V_1(t, x; k_M^*) = V_1^*(t, x) \leq V_1(t, x; k)$, for $t \in T$, $x \in \mathbb{R}^n$, and k an admissible control law. Clearly, $M(t, x) \geq V_1^*(t, x)$.

An MCV control law $k_{V|M}^*$ satisfies $V_2(t, x; k_{V|M}^*) = V_2^*(t, x) \leq V_2(t, x; k)$, for $t \in T$, $x \in \mathbb{R}^n$, whenever k is admissible. The corresponding minimal cost variance is given by $V^*(t, x) = V_2^*(t, x) - M^2(t, x)$ for $t \in T$, $x \in \mathbb{R}^n$.

Here we present the full-state-feedback solution of the MCV control problem for a linear system and a quadratic cost function. For full derivation, refer to [27, 34].

We assume that:

$$\sigma(t, x) = E(t), \quad (20)$$

$$L(t, x, k(t, x)) = x(t)'Qx(t) + k'(t, x)R(t)k(t, x), \quad (21)$$

$$\psi(x(t_F)) = x'(t_F)Q_F x(t_F), \quad (22)$$

and

$$f(t, x, k(t, x)) = A(t)x(t) + B(t)k(t, x). \quad (23)$$

Then the linear optimal MCV controller is given by

$$k_{V|M}^*(t, x) = -R^{-1}(t)B'(t)[\mathcal{M}(t) + \gamma(t)\mathcal{V}(t)]x,$$

where $\mathcal{M}$ and $\mathcal{V}$ are the solutions of the coupled Riccati equations (suppressing the time argument):

$$\dot{\mathcal{M}} + A'\mathcal{M} + \mathcal{M}A + Q - \mathcal{M}BR^{-1}B'\mathcal{M} + \gamma^2\mathcal{V}BR^{-1}B'\mathcal{V} = 0 \quad (24)$$

and

$$\dot{\mathcal{V}} + 4\mathcal{M}EWE'\mathcal{M} + A'\mathcal{V} + \mathcal{V}A - \mathcal{M}BR^{-1}B'\mathcal{V} - \mathcal{V}BR^{-1}B'\mathcal{M} - 2\gamma\mathcal{V}BR^{-1}B'\mathcal{V} = 0, \quad (25)$$

with boundary conditions $\mathcal{M}(t_F) = Q_F$ and $\mathcal{V}(t_F) = 0$.

4. Conclusions and Future Research Topics

A survey of the historical underpinnings of risk-sensitive control theory has been presented, together with some of its relationships to the notion of optimal control cost cumulants. Although the results presented are mostly theoretical, there also exists some applications orientred results. For RS controlled missile guidance example see [29], and for RS controlled structure application see [33, 34]. Another interesting as well as a little surprising are of application is in the field of economics. For MCV applications see [34]. Nevertheless, much more research is needed in applying RS and MCV control, expecially in the nonlinear case. Along the line of theory, there are some results in full state feedback RS control [10]. If one views RS control as an optimization of infinite sum of cumulants, with LQG as an optimization of the first cumulant, MCV as an optimization of the second cumulant, and so forth, then optimization of any cumulant (cost cumulant control) is a possible extension and a generalization of LQG, MCV, and RS control theory.

REFERENCES

[1] T. Başar and P. Bernhard, *H_∞ Optimal Control and Related Minimax Design Problems.* Boston: Birkhäuser, 1991.

[2] A. Bensoussan and J. H. van Schuppen, "Optimal Control of Partially Observable Stochastic Systems with an Exponential-of-Integral Performance Index," *SIAM Journal on Control and Optimization,* Volume 23, pp. 599–613, 1985.

[3] A. Bensoussan, *Stochastic Control of Partially Observable Systems,* London: Cambridge University Press, 1992.

[4] A. Bensoussan and R. J. Elliott, "A Finite-Dimensional Risk-Sensitive Control Problem," *SIAM J. Control and Optimization,* Vol. 33, No. 6, November 1995, pp. 1834–1846.

[5] A. E. Bryson, Jr., and Y.-C. Ho, *Applied and Optimal Control, Optimization, Estimation, and Control,* Revised Printing, New York: Hemisphere Publishing Co., 1975

[6] M. C. Campi and M. R. James, "Risk-Sensitive Control: A Bridge between H_2 and H_∞ Control," *Proceedings of the 32nd Conference on Decision and Control,* San Antonio, Texas, December 1993.

[7] M. H. A. Davis, *Linear Estimation and Stochastic Control,* London: Halsted Press, 1977.

[8] W. H. Fleming and R. W. Rishel, *Deterministic and Stochastic Optimal Control.* New York: Springer-Verlag, 1975.

[9] W. H. Fleming and H. M. Soner, *Controlled Markov Processes and Viscosity Solutions.* New York: Springer-Verlag, 1992.

[10] W. H. Fleming and W. M. McEneaney, "Risk-Sensitive Control on an Infinite Time Horizon," *SIAM J. Control and Optimization,* Vol. 33, No. 6, November 1995.

[11] K. Glover and J. C. Doyle, "State-Space Formulae for All Stabilizing Controllers that Satisfy an H_∞-Norm Bound and Relations to Risk Sensitivity," *Systems and Control Letters*, Volume 11, pp. 167–172, 1988.

[12] K. Glover, "Minimum Entropy and Risk-Sensitive Control: The Continuous Time Case," *Proceedings 28th IEEE Conference on Decision and Control*, pp. 388–391, December 1989.

[13] D. H. Jacobson, "Optimal Stochastic Linear Systems with Exponential Performance Criteria and Their Relationship to Deterministic Differential Games," *IEEE Transactions on Automatic Control*, AC-18, pp. 124–131, 1973.

[14] M. R. James, J. S. Baras, and R. J. Elliott, "Risk-Sensitive Control and Dynamic Games for Partially Observed Discrete-Time Nonlinear Systems," *IEEE Transactions on Automatic Control*, AC-39, No. 4, pp. 780–792, 1994.

[15] P. R. Kumar and J. H. van Schuppen, "On the Optimal Control of Stochastic Systems with an Exponential-of-Integral Performance Index," *Journal of Mathematical Analysis and Applications*, Volume 80, pp. 312–332, 1981.

[16] H. Kushner *Introduction to Stochastic Control*, New York: Holt,Rinehart and Winston, Inc., 1971.

[17] H. Kwakernaak and R. Sivan, *Linear Optimal Control Systems*, John Wiley & Sons, Inc., New York, 1972.

[18] S. R. Liberty, "Characteristic Functions of LQG Control," Ph.D. Dissertation, Department of Electrical Engineering, University of Notre Dame, August 1971.

[19] S. R. Liberty and R. C. Hartwig, "On the Essential Quadratic Nature of LQG Control-Performance Measure Cumulants," *Information and Control*, Volume 32, Number 3, pp. 276–305, 1976.

[20] S. R. Liberty and R. C. Hartwig, "Design-Performance-Measure Statistics for Stochastic Linear Control Systems," *IEEE Transactions on Automatic Control*, AC-23, Number 6, pp. 1085–1090, December 1978.

[21] T. Runolfsson, "The Equivalence between Infinite-Horizon Optimal Control of Stochastic Systems with Exponential-of-Integral Performance Index and Stochastic Differential Games," *IEEE Transactions on Automatic Control*, Vol. 39, No. 8, pp. 1551–1563, 1994.

[22] M. K. Sain, "On Minimal-Variance Control of Linear Systems with Quadratic Loss," *Ph.D Thesis*, Department of Electrical Engineering and Coordinated Science Laboratory, University of Illinois, Urbana, Illinois, January 1965.

[23] M. K. Sain, "Control of Linear Systems According to the Minimal Variance Criterion—A New Approach to the Disturbance Problem," *IEEE Transactions on Automatic Control*, AC-11, No. 1, pp. 118–122, January 1966.

[24] M. K. Sain and C. R. Souza, "A Theory for Linear Estimators Minimizing the Variance of the Error Squared," *IEEE Transactions on Information Theory*, IT-14, Number 5, pp. 768–770, September 1968.

[25] M. K. Sain and S. R. Liberty, "Performance Measure Densities for a Class of LQG Control Systems," *IEEE Transactions on Automatic Control*, AC-16, Number 5, pp. 431–439, October 1971.

[26] M. K. Sain, Chang-Hee Won, and B. F. Spencer, Jr. "Cumulant Minimization and Robust Control," *Stochastic Theory and Adaptive Control, Lecture Notes in Control and Information Sciences 184*, T. E. Duncan and B. Pasik-Duncan (Eds.), Springer-Verlag, pp. 411-425, 1992.

[27] M. K. Sain, C.-H. Won, and B. F. Spencer, "Cumulants in Risk-Sensitive Control: The Full-State-Feedback Cost Variance Case," *Proceedings of 34th CDC*, New Orleans, 1995.

[28] J. L. Speyer, J. Deyst, and D. H. Jacobson, "Optimization of Stochastic Linear Systems with Additive Measurement and Process Noise Using Exponential Performance Criteria," *IEEE Transactions on Automatic Control*, AC-19, No. 4, pp. 358–366, August 1974.

[29] J. L. Speyer, "An Adaptive Terminal Guidance Scheme Based on an Exponential Cost Criterion with Application to Homing Missile Guidance," *IEEE Transactions on Automatic Control*, AC-21, pp. 371–375, 1976.

[30] F. Tung, "Linear Control Theory Applied to Interplanetary Guidance," *IEEE Transactions on Automatic Control*, AC-9, Number 1, pp. 82–89, January 1964.

[31] P. Whittle, *Risk Sensitive Optimal Control*. New York: John Wiley & Sons, 1990.

[32] P. Whittle, "A Risk-Sensitive Maximum Principle: The Case of Imperfect State Observation," *IEEE Transactions on Automatic Control*, Vol. 36, No. 7, pp. 793–801, July 1991.

[33] C.-H. Won, M. Sain, B. Spencer, Jr., "Performance and Stability Characteristics of Risk-Sensitive Controlled Structures Under Seismic Disturbances," *Proceedings American Control Conference*, Seattle, Washington, pp. 1926-1930, June 21, 1995.

[34] C.-H. Won, "Cost Cumulants in Risk-Sensitivity and Minimal Cost Variance Control," Ph. D. Dissertation, University of Notre Dame, Notre Dame, IN, USA, 1995.

[35] E. Wong and B. Hajek, *Stochastic Processes in Engineering Systems*, New York: Springer-Verlag, 1985.