Proceedings of the 11th KACC, October 1996

Failure Recoverability by Exploiting Kinematic Redundancy

Jonghoon Park*, Wan-Kyun Chung**, Youngil Youm**

* Graduate Student in Dept. of Mech. Eng, POSTECH
** Professors in Dept. of Mech. Eng, POSTECH
Tel:+82-562-279-2844; Fax: +82-562-279-5899; E-mail: jhpark@risbot.rist.re.kr

Abstracts This paper is concerned with how to utilize kinematic redundancy to reconstruct the inverse kinematic solution which is not attainable dur to hardware limitations. By analyzing the error due to hardware limitations, we are to show that the recoverability of limitation reduces to the solvability of a reconstruction equation under the feasibility condition. It will be next shown that the reconstruction equation is solvable if the configuration is not a joint-limit singularity. The reconstruction method will be proposed based on the geometrical analysis of recoverability of hardware limitations. The method has the feature that no task motion error is induced by the hardware limitations while minimizing a possible null motion error, under the recoverability assumed.

1. Introduction

Seen from the kinematic viewpoint, the manipulator is represented by a nonlinear map called the *forward kinematics*. To proceed, we assume that the manipulator has n DOF and is to operate in a $m(\leq n)$ dimensional task space. A task position is denoted by $\boldsymbol{p} = (p_1, \cdots, p_m)^T \in \Re^m$ and a pose of the manipulator is specified by $\boldsymbol{q} = (q_1, \cdots, q_n)^T \in \Re^n$. The forward kinematics can be represented of the form

$$\boldsymbol{p} = \boldsymbol{f}(\boldsymbol{q}) \tag{1}$$

where $\boldsymbol{f}$ is a smooth nonlinear map. To operate a task specified in the task space, we have to solve the inverse kinematics of the manipulator, i.e. to find $\boldsymbol{q}$ such that $\boldsymbol{f}(\boldsymbol{q}) = \boldsymbol{p}$ for a given $\boldsymbol{p}$. Since the nonlinear map inversion is not so tractable, the usual inverse kinematics is done at the velocity level under the name of *resolved motion rate control*

$$\dot{\boldsymbol{p}} = \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}}, \quad \boldsymbol{J}(\boldsymbol{q}) = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{q}^T} \tag{2}$$

where the $m \times n$ matrix $\boldsymbol{J}$ is the Jacobian. From (2) the inverse kinematics is solved by inverting the Jacobian matrix, if the pseudoinverse may be taken as a substitute for $\boldsymbol{J}^{-1}$.

The difficulty due to kinematic and hardware limitations lies in solving the inverse kinematics of the manipulator. Most kinematic limitations are manifested as singularity of the Jacobian. This kind of kinematic limitations can be overcome by utilizing kinematic redundancy, i.e. $n > m$. The number $r = n - m$ is referred to as the *degrees of redundancy*. The (kinematically) redundant manipulator is defined as a manipulator which has more degrees of freedom (DOF) than the dimension of the task space. The advantage to overcome kinematic limitations is derived from the fact that there are $(n - m)$ dimensional inverse kinematic solution manifold, called the *self-motion manifold* [1], corresponding to a fixed task position $\boldsymbol{p} \in \Re^m$.

Besides the kinematic limitations, hardware limitations make the inverse kinematics difficult. For example, the inverse kinematic solution which is impossible due to joint travel and/or velocity limits entails a big error. Recently some works were proposed to exploit the kinematic redundancy to solve the hardware limitations. Cheng et al. [2] developed their own inverse kinematic algorithm using the quadratic programming for redundant manipulators which can incorporate the joint travel and velocity limits. Sung et al. [6] also developed their own using the Kuhn-Tucker optimization theorem. They can be considered as a direct method in the sense that the inequality constraint regarding the hardware limitations is directly incorporated in constructing the inverse kinematic solution. That is, they tried to solve directly the inequality and equality constrained nonlinear optimization problem:

To find a solution $\boldsymbol{q}$ or $\dot{\boldsymbol{q}}$
which optimizes $m(\boldsymbol{q})$
subject to

$$\boldsymbol{f}(\boldsymbol{q}) = \boldsymbol{p} \quad \text{or} \quad \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}} = \dot{\boldsymbol{p}}$$

$$\underline{\boldsymbol{q}} \leq \boldsymbol{q} \leq \overline{\boldsymbol{q}}$$
$$\underline{\dot{\boldsymbol{q}}} \leq \dot{\boldsymbol{q}} \leq \overline{\dot{\boldsymbol{q}}}.$$

Recently, there appeared a method to solve the hardware limitations called the joint velocity reconstruction method [5]. The basic intuition behind the method is that the task motion error induced by the hardware limitation can be eliminated by adjusting the velocity of the other joints which are not limited. Inspired by the method, we refine the properties pertinent to reconstructing a limited joint velocity using the notion of the recoverability. The concept of the recoverability will prove useful in understanding the joint velocity reconstruction.

2. Weighted Kinematically–Decoupled Kinematic Modeling

In this section, a brief review of the kinematically decoupled joint space decomposition of redundant manipulators [4], focusing on the aspects concerned with the inverse kinematics will be given. A main contribution of the method lies in providing a systematic method to define a coordinate transformation to the kinematically decoupled coordinates which consists of the task velocity and minimally parametrized null velocity. Also, the method was extended to the weighted pseudoinverse method [4]. Since the weighted pseudoinverse method is more general, we discuss the latter.

The fact that a weighted pseudoinverse of $\boldsymbol{J}$, denoted by $\boldsymbol{J}^{W+}$, induces a joint space decomposition is understood by observing the inverse kinematics

$$\dot{q} = \boldsymbol{J}^{W+}\dot{\boldsymbol{p}} + \left(\boldsymbol{I} - \boldsymbol{J}^{W+}\boldsymbol{J}\right) z. \tag{3}$$

That is, the inverse kinematic (velocity) solution space is a direct sum of the range space of $\boldsymbol{J}^{W+}$ and the weighted null space of $\boldsymbol{J}$. By noting that each space is a vector space of dimension m and r, we can parametrize elements of each subspace with m and r parameters. A minimal parametrization should be followed by a well-defined coordinate transformation. It was shown that the coordinate transformation is given as

$$\begin{aligned} \dot{q} \;=\; & \boldsymbol{W}^{-\frac{1}{2}}\left[\boldsymbol{R}_{W^{-\frac{1}{2}}}\left(\boldsymbol{J}_{W^{-\frac{1}{2}}}\boldsymbol{R}_{W^{-\frac{1}{2}}}\right)^{-1}\right. \\ & \left.\left|\boldsymbol{N}_{W^{-\frac{1}{2}}}\left(\boldsymbol{Z}_{W^{-\frac{1}{2}}}\boldsymbol{N}_{W^{-\frac{1}{2}}}\right)^{-1}\right]\begin{pmatrix} \dot{\boldsymbol{p}} \\ \dot{\boldsymbol{n}}_{W^{\frac{1}{2}}} \end{pmatrix}\end{aligned} \tag{4}$$

where the various matrices will be defined below. Note that the minimally parametrized null velocity $\dot{\boldsymbol{n}}_{W^{\frac{1}{2}}} \in \Re^r$, called the weighted null velocity, is defined by

$$\dot{\boldsymbol{n}}_{W^{\frac{1}{2}}} = \boldsymbol{Z}\boldsymbol{W}z. \tag{5}$$

The square matrix in (4) is indeed nonsingular, if $\boldsymbol{J}$ has full column rank m, and defines a coordinate transformation.

The various matrices appearing in (4) can be given using the weighted Jacobian $\boldsymbol{J}_{W^{-\frac{1}{2}}} \in \Re^{m\times n}$ and its null space basis matrix $\boldsymbol{Z}_{W^{-\frac{1}{2}}} \in \Re^{r\times n}$. They are counterparts of $\boldsymbol{J}$ and $\boldsymbol{Z}$ defined in [5] in pseudoinverse decomposition. Using $\boldsymbol{J}(q)$, $\boldsymbol{Z}(q)$, and the square-root of $\boldsymbol{W} \in \Re^{n\times n}$

$$\boldsymbol{W} = \boldsymbol{W}^{\frac{1}{2}}\boldsymbol{W}^{\frac{1}{2}}$$

$\boldsymbol{J}_{W^{-\frac{1}{2}}}$ and $\boldsymbol{Z}_{W^{-\frac{1}{2}}}$ are computed by

$$\boldsymbol{J}_{W^{-\frac{1}{2}}} = \boldsymbol{J}\boldsymbol{W}^{-\frac{1}{2}} \quad \boldsymbol{Z}_{W^{-\frac{1}{2}}} = \boldsymbol{Z}\boldsymbol{W}^{\frac{1}{2}}. \tag{6}$$

By taking the eigenvalue decomposition of a real symmetric matrix $\boldsymbol{J}^T_{W^{-\frac{1}{2}}}\boldsymbol{J}_{W^{-\frac{1}{2}}}$ we get two matrices $\boldsymbol{R}_{W^{-\frac{1}{2}}} \in \Re^{n\times m}$ and $\boldsymbol{N}_{W^{-\frac{1}{2}}} \in \Re^{n\times r}$ from the expression

$$\begin{aligned}\boldsymbol{J}^T_{W^{-\frac{1}{2}}}\boldsymbol{J}_{W^{-\frac{1}{2}}} \;=\; & \left[\begin{array}{cc}\boldsymbol{R}_{W^{-\frac{1}{2}}} & \boldsymbol{N}_{W^{-\frac{1}{2}}}\end{array}\right]\left[\begin{array}{cc}\boldsymbol{\Sigma}^2 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0}\end{array}\right] \\ & \cdot\left[\begin{array}{cc}\boldsymbol{R}_{W^{-\frac{1}{2}}} & \boldsymbol{N}_{W^{-\frac{1}{2}}}\end{array}\right]^T,\end{aligned} \tag{7}$$

where $\Sigma^2 \in \Re^{m\times m}$ is the diagonal matrix with the nonzero eigenvalues of $\boldsymbol{J}^T_{W^{-\frac{1}{2}}}\boldsymbol{J}_{W^{-\frac{1}{2}}}$ as the diagonal elements.

Remark 2..1 *The matrices $\boldsymbol{R}_{W^{-\frac{1}{2}}}$ and $\boldsymbol{N}_{W^{-\frac{1}{2}}}$ can be arbitrary while satisfying (7). However, the arbitrariness does not affect the derivation of this paper.*

The transformation (4) can be considered as a kind of inverse transformation. By inverting the transformation we get the forward transformation from $\dot{q}$ to the kinematically decoupled velocity $\left(\dot{\boldsymbol{p}}^T, \dot{\boldsymbol{n}}^T_{W^{\frac{1}{2}}}\right)^T$

$$\begin{pmatrix} \dot{\boldsymbol{p}} \\ \dot{\boldsymbol{n}}_{W^{\frac{1}{2}}} \end{pmatrix} = \left[\begin{array}{c} \boldsymbol{J}(q) \\ \boldsymbol{Z}(q)\boldsymbol{W}(q) \end{array}\right]\dot{q}. \tag{8}$$

Given $\dot{q}$ at q, the task velocity $\dot{\boldsymbol{p}}$ is obtained by applying $\boldsymbol{J}$, whereas the weighted null velocity $\dot{\boldsymbol{n}}_{W^{\frac{1}{2}}}$ is by $\boldsymbol{Z}\boldsymbol{W}$.

Since (4) is the inverse transformation, the equation can be directly used as an inverse kinematics algorithm. In other words, given a desired task velocity $\dot{\boldsymbol{p}}_d \in \Re^m$, the kinematically decoupled inverse kinematic method constructs the inverse kinematic velocity by

$$\begin{aligned} \dot{q}_d \;=\; & \boldsymbol{W}^{-\frac{1}{2}}\left[\boldsymbol{R}_{W^{-\frac{1}{2}}}\left(\boldsymbol{J}_{W^{-\frac{1}{2}}}\boldsymbol{R}_{W^{-\frac{1}{2}}}\right)^{-1}\right. \\ & \left.\left|\boldsymbol{N}_{W^{-\frac{1}{2}}}\left(\boldsymbol{Z}_{W^{-\frac{1}{2}}}\boldsymbol{N}_{W^{-\frac{1}{2}}}\right)^{-1}\right]\begin{pmatrix} \dot{\boldsymbol{p}}_d \\ \dot{\boldsymbol{n}}_{W^{\frac{1}{2}},d} \end{pmatrix}\end{aligned} \tag{9}$$

where $\dot{\boldsymbol{n}}_{W^{\frac{1}{2}},d}$ is an r-dimensional desired weighted null velocity to utilize kinematic redundancy. The solution by (9) leads to

$$\boldsymbol{J}(q)\dot{q}_d = \dot{\boldsymbol{p}}_d \tag{10}$$

$$\boldsymbol{Z}(q)\boldsymbol{W}(q)\dot{q}_d = \dot{\boldsymbol{n}}_{W^{\frac{1}{2}},d}. \tag{11}$$

Thus, the set of inverse kinematic solutions is parametrized by a r-dimensional null velocity. If $\dot{\boldsymbol{n}}_{W^{\frac{1}{2}},d}$ is obtained from $z \in \Re^n$ in (3) by (5), then two equation (3) and (9) are equivalent.

3. Analysis of Joint Hardware Limitations

When the calculated joint velocity by (3) or (9) can not be achieved within the joint limitations of the following form

$$\underline{\dot{q}} \leq \dot{q}(t_k) \leq \bar{\dot{q}}, \tag{12}$$

the expected performance by (10) and (11) cannot be anticipated.

To proceed the development, let us define the followings.

DEFINITION 3..1 **(degrees of limitation)** The degrees of limitation s is the number of joints which does not satisfy

$$\underline{\dot{q}}_k \leq \dot{q}_k \leq \bar{\dot{q}}_k.$$

Such joint is called the limited joint, and the others are called the feasible joint.

DEFINITION 3..2 **(recoverable)** A hardware limitation is said to be recoverable at q, if there exists, at least one, another $\dot{q}$ within feasible velocity range which produces the same task velocity. Then the velocity is called the reconstructed velocity. Any limitations occurring at a kinematic singularity is not recoverable by definition.

From the definition, we can have the following proposition.

PROPOSITION 3..1 *No velocity limitations are recoverable for nonredundant manipulators.*

The proof is not difficult, as the theorem itself is trivial by understanding that nonredundant manipulators is not recoverable at any q if joint limitation occurs. The actual proof is deferred later. By the definition, if a joint velocity limitation is recoverable at q, then we can choose a reconstructed velocity (within feasible range) as the inverse kinematic solution, since it leads to the same task velocity.

Now the error due to joint limitations will be analyzed. Assume that the degrees of limitation is s, and, say, $k_1, k_2 \cdots k_s$ joints are limited. A motion resulting from the limited solution $\hat{\dot{q}}$

$$\hat{\dot{q}} = (\dot{q}_1 \quad \cdots \quad \hat{\dot{q}}_{k_1} \quad \cdots \quad \dot{q}_j \quad \cdots \quad \hat{\dot{q}}_{k_s} \quad \cdots \quad \dot{q}_n)^T$$

is obtained by

$$\begin{aligned} \hat{\dot{p}} &= J(q)\hat{\dot{q}} \\ \hat{\dot{n}}_{W^{\frac{1}{2}}} &= Z(q)W(q)\hat{\dot{q}}. \end{aligned}$$

Hence it is easy to see that the task motion error is given by

$$\Delta\dot{p} = \dot{p} - \hat{\dot{p}} = J(q)\dot{q} - J(q)\hat{\dot{q}} = \hat{J}(q)\Delta\dot{q} \quad (13)$$

where s-vector $\Delta\dot{q}$ and $m \times s$-matrix $\hat{J}(q)$ is composed as follows

$$\begin{aligned} \Delta\dot{q} &= \begin{pmatrix} \dot{q}_{k_1} - \hat{\dot{q}}_{k_1} \\ \vdots \\ \dot{q}_{k_s} - \hat{\dot{q}}_{k_s} \end{pmatrix} \\ \hat{J}(q) &= \left[\, J^{(k_1)} \quad J^{(k_2)} \quad \cdots \quad J^{(k_s)} \,\right] \end{aligned}$$

where $J^{(i)}$ denotes the j-th column of $J(q)$. Also, the null motion error is similarly obtained

$$\Delta\dot{n}_{W^{\frac{1}{2}}} = Z(q)\widehat{W}(q)\Delta\dot{q} \quad (14)$$

by observing that $\widehat{ZW} = Z\widehat{W}$, where $\widehat{W}$ is composed of k_i-th column of W.

Remark 3..1 *When* $W = I$, $\Delta\dot{n} = \hat{Z}(q)\Delta\dot{q}$, *where* $\hat{Z} \in \Re^{r\times s}$ *consists of* k_i*-th column of* Z.

Note that the joint limitation $\Delta\dot{q}$ induces an error in both task and (weighted) null motion.

Now we are to propose a condition guaranteeing the recoverability of joint limitations using the above error analysis. The basic method we aim at is that we can eliminate the task motion error by adjusting the velocity of the feasible joints. To develop the idea, the adjusting velocity of the following form

$$\tilde{\dot{q}}_0 = (\tilde{\dot{q}}_1 \quad \cdots \quad \underset{\substack{\Uparrow \\ k_1}}{0} \quad \cdots \quad \tilde{\dot{q}}_j \quad \cdots \quad \underset{\substack{\Uparrow \\ k_s}}{0} \quad \cdots \quad \tilde{\dot{q}}_n)^T$$

is considered reflecting the fact that the limited joint is not adjusted. If the limitation is to be recoverable by choosing a candidate reconstructed solution by

$$\dot{q}^* = \hat{\dot{q}} + \tilde{\dot{q}}_0, \quad (15)$$

it should satisfy

$$\dot{p} = J\dot{q}^* = J\hat{\dot{q}} + J\tilde{\dot{q}}_0.$$

In view of

$$J\tilde{\dot{q}}_0 = \tilde{J}\tilde{\dot{q}}$$

where $\tilde{\dot{q}} \in \Re^{n-s}$ consists of the adjusting velocity at $(n-s)$ feasible joints, and $\tilde{J} \in \Re^{m\times(n-s)}$ consists of $J^{(i)}$ with $i \notin \{k_1, \cdots, k_s\}$, there follows

$$\tilde{J}\tilde{\dot{q}} = \dot{p} - J\hat{\dot{q}} = \Delta\dot{p}. \quad (16)$$

The equation is called the *reconstruction equation*, since if there exists a solution $\tilde{\dot{q}} \in \Re^{n-s}$ satisfying the equation, then the limitation is recoverable, and also a reconstructed solution is given by (15).

The following theorem immediately follows.

THEOREM 3..1 *If there exists a solution to the reconstruction equation (16) given* $\Delta\dot{p}$ *such that* $\dot{q}_j^*$ *is feasible for* $j \neq k_i$*, then the joint limitation is recoverable and a reconstructed solution is given by (15).*

Proof. The requirement of no task motion error is clear by the discussions above. Also, by the assumptions regarding the feasibility, the conclusion follows. ∎

4. Joint–limit Singularity and Recoverability

As seen in Theorem 3..1, the recoverability depends on the solvability of the reconstruction equation (16) which is linear in $\tilde{\dot{q}}$, which, in turn, depends on the property of the $m \times (n-s)$-matrix $\tilde{J}(q)$. Before proceeding, the following definition is made.

DEFINITION 4..1 **(joint–limit singularity)** The configuration q is called the *joint-limit singularity* if the reconstruction equation (16) is not consistent for given $\Delta\dot{p}$.

Then the following lemma follows the definition.

LEMMA 4..1 *If the degrees of limitation* s *is greater than the degrees of redundancy* r*, then the limitation is not recoverable for any* q.

Proof. If

$$r < s \implies n - s < m$$

then the matrix $\widetilde{J}$ can not have full column rank. If $\Delta\dot{p}$ does not belong to the column space of $\widetilde{J}(q)$, then (16) is inconsistent.

Even if $\Delta\dot{p}$ does belong to the column space of $\widetilde{J}(q)$ of rank less than m, the limitation is not recoverable since then the configuration is necessarily kinematically singular. By definition, the kinematic singularity was precluded from being recoverable. To show that the latter case reduces to kinematic singularity, assume that

$$\widehat{J}\Delta\dot{q} = \sum_{k=1}^{n-s} d_k \widetilde{J}^{(k)}.$$

This yields

$$0 = \widehat{J}\Delta\dot{q} - \widetilde{J}d = \begin{bmatrix} \widetilde{J} & \widehat{J} \end{bmatrix} \begin{pmatrix} -d \\ \Delta\dot{q} \end{pmatrix}$$

by arranging $d = (d_1, \cdots, d_{n-s})^T$. In view that the matrix has same row rank to J, and that the null vector of the matrix has independent variables $\Delta\dot{q}$ (i.e. if $\Delta\dot{q}$ is fixed, then d is determined.), then we can see that the null space dimension is same as that of $\Delta\dot{q}$. Since $\Delta\dot{q}$ has s independent elements, the null space of J should have dimension s, which is greater than the normal null space dimension r. Hence, there follows the conclusion that J is singular. ∎

The following has been also proved in proof the above lemma.

COROLLARY 4..1 *The null space of J has the dimension* $\max\{r, s\}$, *if the reconstruction equation (16) is consistent.* ∎

Now it is obvious to see that Proposition 3..1 is an immediate corollary to the above one, since in nonredundant manipulator r is always zero. Hence the degrees of limitation s is greater than r, if nonzero.

Now we are in position to state the following lemma, which states that the solvability of the reconstruction equation depends only on the configuration when a limitation occurs, not the velocity. Noting that the joint limit singularity directly concerns the solvability of the reconstruction equation, there follows

LEMMA 4..2 *The joint-limit singularity depends only on the configuration q when the limitation occurs.*

Remark 4..1 *Note that the recoverability consists of two conditions, i.e. the solvability of the reconstruction equation (16) and the feasibility of the adjusting velocity $\tilde{\dot{q}}$. Hence, the joint velocity, or the joint velocity limitation error $\Delta\dot{q}$, affects the feasibility of $\tilde{\dot{q}}$. That is, the deeper the limitation, the less feasible the adjusting velocity. In terms of joint-limit singularity, we can say that the limitation is recoverable at configurations only if the configuration is not joint-limit singular.*

Based on the lemma, the joint-limit singularity can be analytically characterized by the configuration dependent criteria as in [5].

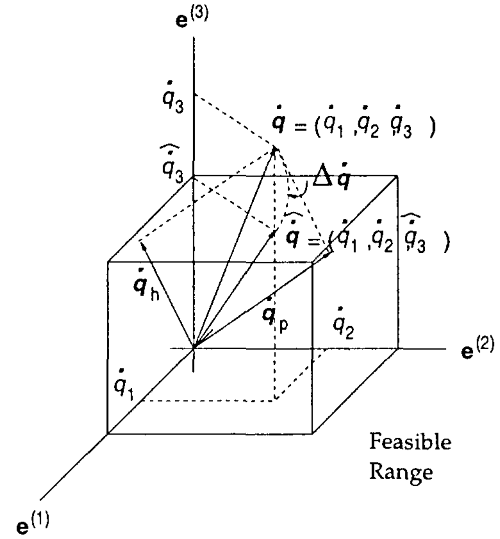

Figure 1: Limitation analysis in joint coordinate

5. Geometric Analysis of Joint Limitations

To grab a physical interpretation of the method, we present a geometrical analysis for a simple case where $n = 3$ and $m = 2$, i.e. $r = 1$. Assume that $\dot{q}$ is an inverse kinematic solution without joint limitation being considered, and one joint, say third joint, is limited.

The situation is depicted in Fig. 1 with respect to the standard joint coordinate consisting of $e^{(1)}$, $e^{(2)}$, and $e^{(3)}$, where $e^{(k)}$ is the vector with 1 at k-th element, and zero otherwise. Usually the velocity $\dot{q}$ is obtained as a sum of the particular and homogeneous velocity, i.e.

$$\dot{q} = \dot{q}_p + \dot{q}_h$$

as shown in the figure. A portion of the feasible region is also drawn. Since $\dot{q}_3$ is limited, the closest feasible value of $\hat{\dot{q}}_3$ is taken as a substitute to construct a limited solution $\hat{\dot{q}}$. The error analysis is not so direct, and it is difficult to know how to find a reconstructed solution. Even it is not easily determined whether the limitation is recoverable or not.

Now a new coordinate, called *net/null coordinate*, consisting of $\{r^{(1)}, r^{(2)}, n^{(1)}\}$, is taken, where $r^{(k)}$ $(k = 1, \cdots, m)$ is the k-th column vector of the R matrix, and $n^{(k)}$ $(k = 1, \cdots, r)$ is the k-th column vector of the N matrix. As shown in [4], any joint velocity $\dot{q} = (\dot{q}_1, \cdots, \dot{q}_n)^T$ in the standard coordinate is represented as

$$\dot{q} = \left(\dot{q}_{net}^T, \ \dot{q}_{null}^T\right)^T = \left(\dot{q}_{net,1}, \cdots, \dot{q}_{net,m}, \ \dot{q}_{null,1}, \cdots, \dot{q}_{null,r}\right)^T$$

with respect to the net/null coordinates. They are transformed to each other by

$$\dot{q} = \begin{bmatrix} R(q) & N(q) \end{bmatrix} \begin{pmatrix} \dot{q}_{net} \\ \dot{q}_{null} \end{pmatrix}. \qquad (17)$$

In the above, the m-vector $\dot{q}_{net}$ is called the *net velocity*, and the r-vector $\dot{q}_{null}$ is called the *null velocity*. [1]

[1]Same terms are used to denote $\dot{n}$ and $\dot{q}_{null}$. However, the

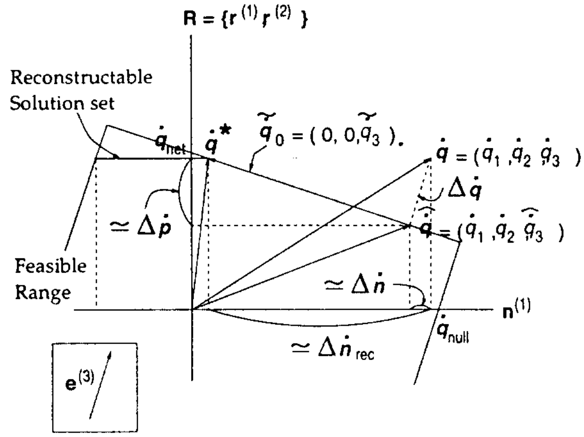

Figure 2: Limitation analysis in kinematically decoupled coordinate

The net and null velocity $\dot{q}_{net}$ and $\dot{q}_{null}$ is in one-to-one correspondence with the task velocity and null velocity in the kinematically decoupled coordinate, respectively, by each nonsingular square coordinate transformation, that is

$$\dot{p} = J(q)R(q)\dot{q}_{net} \qquad \dot{n} = Z(q)N(q)\dot{q}_{null}. \tag{18}$$

By taking a coordinate transformation to the net/null coordinate, the situation in Fig. 1 is represented by Fig. 2. Since the figure shows the direction of $e^{(3)}$, the limited solution $\hat{q}$ is taken along the $e^{(3)}$, as seen in the figure. By definition, any reconstructed solution should have $\dot{q}_{net}$ of $\dot{q}$ as the first m elements in order not to induce a task motion error. In view of this fact, the recoverable solution set is a vector on the lines denoted in the figure. The limited solution $\hat{q}$ is not such, so it induces a task motion error $\Delta\dot{p}$ and also a null motion error $\Delta\dot{n}$. It is not difficult to see that the solution $\tilde{q}_0$ is such that whose direction is the same as the lines defining the boundary of the feasible region, hence the reconstructed solution $\dot{q}^*$ by (16) and (15) is the one located at the intersection of the plane defining a feasible region boundary

$$\dot{q}_3^* = \hat{q}_3$$

and the line defining a set of solutions which does not induce task motion error

$$\dot{q}_{net}^* = \dot{q}_{net}.$$

It should be noted that the reconstructed solution is indeed the one with the minimal null motion error, denoted by $\Delta\dot{n}_{rec}$.

A general case where $r > 1$ and/or the null velocity is parametrized with a weight matrix W can also be understood by similar interpretation. The geometrical construction of the reconstructed solution can be generalized in the multiple redundancy case. We can see that the solution reconstruction is equivalent to finding an intersection of a plane and a line. In this case the problem is reduced to the one regarding an intersection of a r-dimensional hypersurface and a $n-s$-dimensional hypersurface. The r-dimensional hypersurface is define by the constraint equation of $\dot{p} = J\dot{q}$, and in the net/null coordinate it is represented by

$$\dot{q}_{net}^* = \dot{q}_{net}. \tag{19}$$

This surface is called the *inverse kinematics* manifold. The $n-s$ dimensional hypersurface is defined by the feasibility condition, and it is represented with respect to the standard joint coordinates by

$$\dot{q}_j^* = \hat{q}_j, \quad \forall j = k_1, \cdots, k_s \tag{20}$$

and it is called the *feasible boundary* manifold. Two manifolds are well defined. They are indeed hypersurfaces, because each can be represented as a point, line, plane, cube, and so on, depending on the dimension defined above.

However, this geometrical method is not so appealing since it is not systematic. Unfortunately, the method itself can not give a unique solution. To see this, if $r = 2$ and $s = 1$, then the intersection constitutes either a line or a plane in generic cases. A systematic method resolving this kind of multiplicities based on a sense of optimality is additionally required, which is the main subject of the next section.

ambiguity does not arise in the sequel. If one minds, then the latter is properly called the *joint space* null velocity.

6. Reconstruction of Limited Solution

The general reconstruction algorithm parallels the derivations in previous sections. Now assume that the degrees of limitation is s, and k_i-th joint is limited for each $i = 1, \cdots, s$. Then the errors due to the limitation are computed by

$$\Delta\dot{p} = \hat{J}\Delta\dot{q} \qquad \Delta\dot{n}_{W^{\frac{1}{2}}} = Z\widehat{W}\Delta\dot{q}. \tag{21}$$

Also, it is assumed that $r \geq s$ to guarantee recoverability, and the configuration is not joint-limit singular. The recoverability is based on the reconstruction equation rewritten below

$$\tilde{J}\tilde{q} = \Delta\dot{p}. \tag{22}$$

Since $\tilde{J}$ is a $m \times (n-s)$ matrix and it is assumed that $r \geq s$, (22) is consistent. If $r = s$, the equation determines a unique solution $\tilde{q} \in \Re^{n-s}$ by

$$\tilde{q} = \tilde{J}^{-1}\Delta\dot{p}. \tag{23}$$

Hence if the solution by (15) is feasible, then the solution reconstructs the limited solution.

However, when $r > s$, there exist many $\tilde{q}$'s to satisfy the equation. A scheme to choose a unique solution is required and we aim to choose the unique solution which minimizes the null motion error involved in the solution reconstruction. That is, one solution which minimizes $\left\|\Delta\dot{n}_{W^{\frac{1}{2}},rec}\right\|$ is attained as the unique solution. For formulation, the null motion error is

$$\begin{aligned}\Delta\dot{n}_{W^{\frac{1}{2}},rec} &= \dot{n}_{W^{\frac{1}{2}}} - ZW\dot{q}^* \\ &= ZW\dot{q} - ZW\widehat{\dot{q}} - ZW\widetilde{\dot{q}}_0 \\ &= Z\widehat{W}\Delta\dot{q} - Z\widetilde{W}\widetilde{\dot{q}} \\ &= \Delta\dot{n}_{W^{\frac{1}{2}}} - Z\widetilde{W}\widetilde{\dot{q}}. \qquad (24)\end{aligned}$$

Thus the problem to resolve a unique solution to (22) is reduced to the following linear-equality constrained least-square problem

find a solution $\widetilde{\dot{q}}^* \in \Re^{n-s}$
which minimizes

$$\left\|Z\widetilde{W}\widetilde{\dot{q}} - \Delta\dot{n}_{W^{\frac{1}{2}}}\right\| \qquad (25)$$

subject to

$$\widetilde{J}\widetilde{\dot{q}} = \Delta\dot{p}.$$

The solution is well established and one form is found as [3, 5]

$$\begin{aligned}\widetilde{\dot{q}}^* &= \widetilde{J}^+\Delta\dot{p} \qquad (26)\\ &+ \left[Z\widetilde{W}\left(I - \widetilde{J}^+\widetilde{J}\right)\right]^+\left(\Delta\dot{n}_{W^{\frac{1}{2}}} - Z\widetilde{W}\widetilde{J}^+\Delta\dot{p}\right).\end{aligned}$$

All the pseudoinverses used in (26) are those of full row rank matrices, and they can be calculated by

$$A^+ = A^T\left(AA^T\right)^{-1}. \qquad (27)$$

Note that it is solvable if (22) is consistent, and furthermore if rank $\left\{\begin{bmatrix}\widetilde{J} \\ Z\widetilde{W}\end{bmatrix}\right\}$ is $n-s$, then the solution is unique and has minimal length property. Hence if the solution by (15) from (26) is feasible, then the solution reconstructs the limited solution. It is worth commenting that the reconstructed solution is indeed the one which minimized the possible null motion error.

Remark 6..1 *There is a possibility that the reconstructed solution $\dot{q}^*$ by the proposed method happens to be unfeasible, since $\widetilde{\dot{q}}$ may make any of originally feasible joints limited. In this case, the degrees of limitation is increased by the number of newly limited joints. Hence, the calculation should be iterated again. The multiple degrees of redundancy will find usefulness in such cases. Note that the number of iteration is at most r. If it does not find a reconstructed solution within r times of iterations, the limitation is not recoverable. Also, during the iterations, all the matrices J, Z, W, and $R_{W^{-\frac{1}{2}}}$, and $N_{W^{-\frac{1}{2}}}$ are fixed, whereas a number of clumns of the matrices $\widehat{J}$, $\widetilde{J}$, $\widehat{W}$, and $\widetilde{W}$ are added or eliminated to and from each matrix. We propose the following step-by-step algorithm:*

i : calculate $\dot{q}$ by (9), and set $\dot{q}^* = \dot{q}$
ii-a : if $\dot{q}^*$ is not limited, go to step i.
ii-b : else do the followings:
iii-a : calculate d and f
iii-b : solve $\widetilde{\dot{q}}^*$ by (26)
iii-c : reconstruct $\dot{q}^*$ by (15)
iii-d : go to step ii-a.

The numerical simulations according to the above algorithm can be found in [5].

7. Conclusions

In this paper, we analyzed the general properties of joint hardware limitations for redundant manipulators. Two most important concepts introduced are the *recoverability*, and the *joint-limit singularity*. Next, we proposed a reconstruction method by solving the reconstruction equation (16) and adjusting the feasible joints by (15). Note that if the joint configuration is not joint–limit singular, then the joint limitation is recoverable if the adjusting velocity is feasible. The method motivated by the failure recoverability results in the unique solution minimizing possible null motion error.

References

[1] Burdick, J. W., "On the inverse kinematics of redundant manipulators: Characterization of the self-motion manifold", in *Proc. 1989 IEEE Int. Conf. on Robotics and Automation*, pp.264–270, 1989

[2] Cheng, F.-T., T.-H. Chen, and Y.-Y. Sun, "Resolving manipulator redundancy under inequality constraints", in *IEEE Trans. on Robotics and Automation*, vol. 10, no. 1, pp. 65–71, 1994

[3] Lawson, C. L.,, and R. J. Hanson, *Solving Least Square Problems*, Prentice-Hall, Inc., 1974

[4] Park, J., W.-K. Chung, and Y. Youm, "Weighted decomposition of kinematics and dynamics of kinematically redundant manipulators", in *Proc. 1996 IEEE Int. Conf. on Robotics and Automation*, pp. 480–486, 1996

[5] Park, J., W.-K. Chung, Y. Youm, and M. Kim, "Reconstruction of Inverse Kinematic Solution Subject to Joint Kinematic Limits Using Kinematic Redundancy", will appear in *Proc. 1996 IEEE/RSJ Int. Conf. on Intelligent Robots and Systems*, 1996

[6] Sung, Y. W., D. K. Cho, M. J. Chung, and K. Koh, "A constraints-based method of the inverse kinematics for redundant manipulators", in *Proc. IEEE/RSJ/GI Int. Conf. on Intelligent Robots and Systems*, pp. 531–538, 1994