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Resolution of Kinematic Redundancy Using Constrained Optimization Techniques under Kinematic Inequality Constraints

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Abstracts This paper considers a global resolution of kinematic redundancy under inequality constraints as a constrained optimal control. In this formulation, joint limits and obstacles are regarded as state variable inequality constraints, and joint velocity limits as control variable inequality constraints. Necessary and sufficient conditions are derived by using Pontryagin's minimum principle and penalty function method. These conditions leads to a two-point boundary-value problem (TPBVP) with natural, periodic and inequality boundary conditions. In order to solve the TPBVP and to find a global minimum, a numerical algorithm, named two-stage algorithm, is presented. Given initial joint pose, the first stage finds the optimal joint trajectory and its corresponding minimum performance cost. The second stage searches for the optimal initial joint pose with globally minimum cost in the self-motion manifold. The effectiveness of the proposed algorithm is demonstrated through a simulation with a 3-*dof* planar redundant manipulator.
Keywords Redundant manipulator, Inverse kinematics, Global optimization, Kinematic control, Pontryagin's minimum principle, Penalty function method, Evolutionary programming

1 Introduction

Robot manipulators often show kinematical limitations in the face of singularities, joint limits, joint velocity limits, obstacles in the task space, etc. To overcome these limitations and to increase dexterity, the usage of kinematic redundancy has been widely studied.

So far, these problems have been defined by using optimization theory which minimizes a performance measure subject to equality and inequality constraints. Such approaches to resolve kinematic redundancy can be classified into two main categories: local resolution[1, 2] and global one. This paper focuses on global optimization approaches.

For this purpose, many researchers[3, 4] have taken advantage of the well-known calculus of variation approach and Pontryagin's minimum principle[5] with equality constraints. However, relatively little has been done to incorporate the *inequality constraints* into the global redundancy resolution.

Recent work[6] has proposed one such method treating joint space inequality constraints and task space ones by using a global exterior penalty function method. It was pointed out[6] that the goal was to find a feasible path rather than an optimal one. In addition, it appeared that the method considered only configuration-dependent inequality constraints.

The main approach of this paper is to regard a global resolution of kinematic redundancy under inequality constraints as an optimal control with state and control variable inequality constraints. Hence this approach addresses virtually all of the kinematic inequality constraints and if successful, guarantees the cyclic behavior of joint motion by using periodic boundary conditions.

2 Optimal Formulations

Our formulation begins with the integral type performance measure

$$Q = \int_{t_0}^{t_f} G(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}, t)dt \quad (1)$$

subject to the kinematic equality constraints

$$\boldsymbol{x}(t) = \boldsymbol{f}(\boldsymbol{\theta}(t)) \quad (2)$$

or

$$\dot{\boldsymbol{x}}(\boldsymbol{\theta}(t)) = \boldsymbol{J}(\boldsymbol{\theta}(t))\dot{\boldsymbol{\theta}}(t), \quad (3)$$

and the kinematic inequality constraints

$$\boldsymbol{r}(\boldsymbol{\theta}(t), \dot{\boldsymbol{\theta}}(t)) \leq 0 \quad (4)$$

where $\boldsymbol{\theta}(t) \in \Re^n$ is the joint vector, $\boldsymbol{x}(t) \in \Re^m$ denotes the vector for position and orientation of the end effector, $\boldsymbol{f}$ is an m-dimensional vector, and $\boldsymbol{J} \in \Re^{m \times n}$ with $m < n$ denotes a Jacobian matrix.

As a reasonable candidate for $G(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}, t)$, many researchers have chosen the following function:

$$G(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}, t) = \frac{1}{2}\dot{\boldsymbol{\theta}}^T \boldsymbol{W}(\boldsymbol{\theta}, t)\dot{\boldsymbol{\theta}} + k_h p(\boldsymbol{\theta}, t) \quad (5)$$

where $\boldsymbol{W}$ denotes a symmetric and positive definite matrix for weighting joint velocity, and $p(\boldsymbol{\theta})$ is a function of joint pose to be minimized, such as the inverse of manipulability measure.

The kinematic inequality constraints deal with joint limits, obstacles, and joint velocity limits in a natural way. Let $\theta_{iu}(\theta_{il})$ denote the upper(lower) joint limit for joint i, then the inequality constraints for joint limits can be described as:

$$r_i = \theta_i - \theta_{iu} \leq 0 \quad (i = 1, \ldots, n) \quad (6)$$

$$r_{n+i} = -\theta_i + \theta_{il} \leq 0 \quad (i = 1, \ldots, n). \quad (7)$$

To formulate obstacles, a kind of distance measure between robot and obstacles can be utilized. In this case, one can consider the following inequality constraints:

$$r_{2n+i} = -(\text{distance measure}) \leq 0 \quad (i = 1, \ldots, q) \quad (8)$$

where q is the number of the obstacle condition.

Similarly, let $\dot{\theta}_{iu}(\dot{\theta}_{il})$ is the upper(lower) joint velocity limit for the i-th joint. Then the inequality constraints for joint velocity limits can be formulated by using the matrix form as follows:

$$A\dot{\boldsymbol{\theta}} - \boldsymbol{b} \leq \mathbf{0} \quad (9)$$

with

$$A = \begin{bmatrix} I_n \\ -- \\ -I_n \end{bmatrix} \in \Re^{2n \times n}, \; \boldsymbol{b} = \begin{bmatrix} \dot{\theta}_{1u} \\ \vdots \\ \dot{\theta}_{nu} \\ -- \\ -\dot{\theta}_{1l} \\ \vdots \\ -\dot{\theta}_{nl} \end{bmatrix} \in \Re^{2n}. \quad (10)$$

Suppose that $\dot{\boldsymbol{\theta}}$ satisfying eq.(3) exists. Then, it can be replaced by

$$\dot{\boldsymbol{\theta}} = J^+\dot{\boldsymbol{x}} + (I_n - J^+J)\boldsymbol{u} \equiv \boldsymbol{g}(\boldsymbol{\theta}, \boldsymbol{u}, t) \quad (11)$$

where $\boldsymbol{u}$ denotes an n-dimensional vector. So, through this formulation, a redundancy resolution problem may be regarded as a kind of optimal control problems with $\boldsymbol{u} \in \Re^n$ as a control variable vector. It is also well known that $\boldsymbol{u}$ is often utilized as a gradient vector of performance measure for local optimization. In the context of optimal control, configuration-dependent inequality constraints such as joint limits and obstacles may be regarded as *state variable inequality constraints*, and joint velocity limits as *control variable inequality constraints.*

So, for the joint-limits and obstacles, one more state variable can be included by using the penalty functions as follows:

$$\dot{\theta}_{n+1} = \sum_{i=1}^{2n+q} \phi_i r_i(\boldsymbol{\theta})^2 H(r_i) = g_{n+1} \quad (12)$$

where ϕ_i is a weighting scalar, and $H(r)$ is a unit Heaviside step function defined by

$$H(r) = \begin{cases} 0 & \text{if } r \leq 0 \\ 1 & \text{otherwise.} \end{cases} \quad (13)$$

To incorporate the state and control variable inequality constraints mentioned above into our formulation, the performance cost can be modified as follows:

$$Q^\star = D(\boldsymbol{\theta}, \theta_{n+1}, t) + \int_{t_0}^{t_1} G^\star(\boldsymbol{\theta}, \boldsymbol{u}, t)dt \quad (14)$$

where D is chosen for boundary conditions, which will be discussed later. $G^\star$ includes the penalty functions for control variable inequality constraints as follows:

$$\begin{aligned} G^\star(\boldsymbol{\theta}, \boldsymbol{u}, t) &= \frac{1}{2}\boldsymbol{g}^T \boldsymbol{W} \boldsymbol{g} + k_h p(\boldsymbol{\theta}) \\ &\quad + \frac{1}{2}(A\boldsymbol{g} - \boldsymbol{b})^T \boldsymbol{K}_t \boldsymbol{H}(A\boldsymbol{g} - \boldsymbol{b}) \end{aligned} \quad (15)$$

where $\boldsymbol{K}_t \in \Re^{2n \times 2n}$ is a constant diagonal matrix with positive elements and $\boldsymbol{H} \in \Re^{2n \times 2n}$ is a diagonal matrix with Heaviside functions as elements, which is an abbreviated form of $\boldsymbol{H}(A\dot{\boldsymbol{\theta}} - \boldsymbol{b})$ for brevity.

According to Pontryagin's minimum principle, the Hamiltonian for this purpose is defined as follows:

$$\mathcal{H}(\boldsymbol{\theta}(t), \boldsymbol{u}(t), \boldsymbol{\psi}(t), t) = G^\star + \boldsymbol{\psi}^T \boldsymbol{g} + \psi_{n+1} g_{n+1} \quad (16)$$

where $\boldsymbol{\psi} \in \Re^n$ and $\psi_{n+1} \in \Re^1$ are costate vectors. For brevity of the formulation, yet without loss of generality, let us assume that $\boldsymbol{W}$ is a constant.

Then, the necessary conditions for the optimality are derived as follows:

$$\dot{\boldsymbol{\theta}} = \left(\frac{\partial \mathcal{H}}{\partial \boldsymbol{\psi}}\right)^T = \boldsymbol{g}(\boldsymbol{\theta}, \boldsymbol{u}, t) \quad (17)$$

$$\dot{\theta}_{n+1} = \frac{\partial \mathcal{H}}{\partial \psi_{n+1}} = g_{n+1} \quad (18)$$

$$\begin{aligned} \dot{\boldsymbol{\psi}} &= -\left(\frac{\partial \mathcal{H}}{\partial \boldsymbol{\theta}}\right)^T \\ &= -\left(\frac{\partial \boldsymbol{g}}{\partial \boldsymbol{\theta}}\right)^T \left(\boldsymbol{W}\boldsymbol{g} + \boldsymbol{\psi} + A^T \boldsymbol{K}_t^T \boldsymbol{H}^T (A\boldsymbol{g} - \boldsymbol{b})\right) \\ &\quad - k_h \left(\frac{\partial p}{\partial \boldsymbol{\theta}}\right)^T - \psi_{n+1} \sum_{i=1}^{2n+q} r_i(\boldsymbol{\theta}) \left(\frac{\partial r_i}{\partial \boldsymbol{\theta}}\right)^T H(r_i) \end{aligned} \quad (19)$$

$$\dot{\psi}_{n+1} = -\frac{\partial \mathcal{H}}{\partial \theta_{n+1}} = 0 \quad (20)$$

and

$$\begin{aligned} \left(\frac{\partial \mathcal{H}}{\partial \boldsymbol{u}}\right)^T &= (I_n - J^+J)\{(\boldsymbol{W} + A^T \boldsymbol{K}_t^T \boldsymbol{H}^T A)\boldsymbol{g} \\ &\quad - A^T \boldsymbol{K}_t^T \boldsymbol{H}^T \boldsymbol{b} + \boldsymbol{\psi}\} = \mathbf{0}. \end{aligned} \quad (21)$$

Let us consider projecting the Hessian matrix $\partial^2\mathcal{H}/\partial \boldsymbol{u}^2$ into the null space of J, which leads to the following positive definite matrix:

$$Z\left(\frac{\partial^2 \mathcal{H}}{\partial \boldsymbol{u}^2}\right)Z^T = Z(\boldsymbol{W} + A^T \boldsymbol{K}_t^T \boldsymbol{H}^T A)Z^T \quad (22)$$

where Z is a null space matrix of J, the relation of which is

$$ZJ^T = \mathbf{0}. \quad (23)$$

Therefore, satisfaction of (21) is sufficient as well as necessary to guarantee that optimal $\boldsymbol{u}$ causes $\mathcal{H}$ to be a minimum.

3 Boundary Conditions

The boundary conditions(BC) as well as the necessary conditions mentioned in the preceding section must be considered to obtain a unique optimal solution satisfying constraints. Among such boundary conditions, three cases are noteworthy: natural boundary conditions, periodic boundary conditions, boundary conditions for state variable inequality constraints.

3.1 Natural Boundary Conditions

These conditions consider the case where there are no requirements on both the initial and the final joint poses except for the kinematic equality constraints (2). These can be obtained with the transversality condition in [3] and are optimally modified with Z in this paper, as is shown in Table 1.

3.2 Periodic Boundary Conditions

Periodic joint motion requires the following periodic boundary conditions

$$\boldsymbol{\theta}(t_0) = \boldsymbol{\theta}(t_f). \quad (24)$$

In order to fix the final value of $\boldsymbol{\theta}$, D is given by

$$D(\boldsymbol{\theta}, t_f) = \frac{1}{2}[\boldsymbol{\theta}(t_f) - \boldsymbol{\theta}(t_0)]^T \boldsymbol{K}_f [\boldsymbol{\theta}(t_f) - \boldsymbol{\theta}(t_0)] \quad (25)$$

where K_f is a constant diagonal matrix with positive elements. So, the extra boundary conditions are derived as

$$\psi(t_f) = \left(\frac{\partial D}{\partial \theta}\right)^T = K_f^T[\theta(t_f) - \theta(t_0)] = 0. \quad (26)$$

3.3 BCs for Inequality Constraints

If and only if the state variable inequality constraints are satisfied, the additional state θ_{n+1} does not change with time. This fact leads to the boundary condition for θ_{n+1} as follows:

$$\theta_{n+1}(t_0) = \theta_{n+1}(t_f). \quad (27)$$

In this case, D is defined as

$$D(\theta_{n+1}, t_f) = \frac{1}{2}k_f[\theta_{n+1}(t_f) - \theta_{n+1}(t_0)]^2 \quad (28)$$

where k_f is a constant scalar. Then, we obtain

$$\psi_{n+1}(t_f) = \frac{\partial D}{\partial \theta_{n+1}} = k_f[\theta_{n+1}(t_f) - \theta_{n+1}(t_0)] = 0. \quad (29)$$

Types	Left End	Right End
Natural BCs	$Z\psi(t_0) = 0$	$x(t_f) = f(\theta(t_f))$ $\psi(t_f) = 0$
Periodic BCs	$x(t_0) = f(\theta(t_0))$	$\theta(t_f) = \theta(t_0)$ $\psi(t_f) = 0$
Inequality BCs		$\theta_{n+1}(t_f) = \theta_{n+1}(t_0)$ $\psi_{n+1}(t_f) = 0$

Table 1: Summary of boundary conditions

The boundary conditions mentioned above are summarized in Table 1. Now, we have a two-point boundary-value problem (TPBVP) which consists of $2n + 2$ first-order differential equations and the same number of boundary conditions.

4 Numerical Algorithm

In this paper, without loss of generality, we consider boundary conditions for periodic joint motion and inequality constraints, and we propose a two-stage algorithm which firstly selects an initial joint pose to find the corresponding minimum cost, and next obtains the optimal joint pose with globally minimum cost.

4.1 First Stage Algorithm

Given the desired end-effector velocity vector $\dot{x}(t)$, initial joint pose $\theta(t_0)$, termination constant ϵ, and maximum number of iteration M, the procedure of the proposed algorithm is as follows:

1. Initialize the control history $u^{(0)}(t)$, $t \in [t_0, t_f]$ by using a discrete approximation which subdivides the interval $[t_0, t_f]$ into N subintervals with equal duration. Consider $u^{(0)}(t)$ as being piecewise-constant during each of these subintervals; i.e.

$$u^{(0)}(t) = u^{(0)}(t_k), \; t \in [t_k, t_{k+1}) \quad (30)$$

   where $k = 0, 1, \dots, N-1$. After setting the iteration index j zero, proceed to step 2.

2. Using the control history $u^{(j)}$, integrate the state equations (17) and (18) from t_0 to t_f with the given initial conditions $\theta(t_0)$ and $\theta_{n+1}(t_0)$ and store the resulting state trajectory $\theta^{(j)}(t)$ and $\theta_{n+1}^{(j)}(t)$.
3. Compute $\psi^{(j)}(t_f)$ and $\psi_{n+1}^{(j)}(t_f)$ by substituting $\theta^{(j)}(t_f)$ and $\theta^{(j)}(t_f)$ into the boundary conditions (26) and (29) Using this value of $\psi^{(j)}(t_f)$ and $\psi_{n+1}^{(j)}(t_f)$ as the initial condition, control history $u^{(j)}$, state trajectory $\theta^{(j)}$ and $\theta_{n+1}^{(j)}$, integrate the costate equations (19) and (20) from t_f to t_0, and compute $Q^\star$ and the following

$$T_c = \int_{t_0}^{t_f} \left\| \frac{\partial \mathcal{H}^{(j)}}{\partial u}(t) \right\|^2 dt. \quad (31)$$

   At each sampling time t_k, generate a new control given by

$$u^{(j+1)}(t_k) = u^{(j)}(t_k) - K_u \frac{\partial \mathcal{H}^{(j)}}{\partial u}(t_k). \quad (32)$$

4. If the terminal condition

$$T_c \le \epsilon \quad \text{or} \quad j \ge M \quad (33)$$

   is satisfied, stop the iterative procedure. Store the resulting optimal state, control and minimum cost $Q^\star$. Otherwise, set $j = j + 1$ and go to step 2.

4.2 Second Stage Algorithm

Let us consider the self-motion manifold as follows:

$$\Gamma(x) = \{\theta \in \Re^n : f(\theta) = x\}. \quad (34)$$

The self-motion manifold can be parameterized with $\alpha \in \Re^{n-m}$, which can span the whole self-motion manifold. Then, we obtain $\theta(t_0) = \theta(\alpha)$.

Given $\theta(t_0)$, global minimum cost $Q^\star_{\min}$ is unique, thus we can consider the following function:

$$Q^\star_{\min} = Q^\star_{\min}(\alpha). \quad (35)$$

Now, the problem is modified to find the global minimum of $Q^\star_{\min}(\alpha)$. This function, however, is such a complicated nonlinear function that it is hard to find a global minimum.

Simply, the conventional search method can be considered, which searches the whole space spanned by α. But this may require heavy computational amounts when the degree of redundancy(*dor*) is more than one. To overcome this problem, more efficient algorithms such as the Evolutionary Programming(EP)[7] may also be considered.

5 Simulation

A 3-*dof* planar robot manipulator is considered as a simulation example because of its simplicity for explanation. The link lengths are $l_1 = 3.0$, $l_2 = 2.5$, and $l_3 = 2.0$ units. The primary task for the end-effector is a cyclic motion (i.e. tracking a circle in a plane of the task space) which is represented as follows:

$$x(t) = \begin{bmatrix} -\cos(2\pi t) + 3.0 \\ -sin(2\pi t) \end{bmatrix} \quad \text{for } t \in [0, 1]. \quad (36)$$

For a configuration-dependent performance measure, the inverse of the well-known manipulability measure is selected as

$$p(\theta) = 1/\sqrt{det(JJ^T)} \quad (37)$$

i	θ_{il} [rad]	θ_{iu} [rad]	$\dot{\theta}_{il}$ [rad/sec]	$\dot{\theta}_{iu}$ [rad/sec]
1	-1.7	1.7	-2.0	2.0
2	-2.1	2.1	-2.0	2.0
3	-2.1	2.1	-2.0	2.0

Table 2: The physical limits of joints and their velocities of the simulated manipulator

and $\boldsymbol{W} = \boldsymbol{I}$ is treated for the weighting matrix.

The kinematic limitation of this robot is shown in Table 2. Then, the seventh inequality constraint for an obstacle in the task space is represented as:

$$r_7(\boldsymbol{\theta}) = -(l_1\cos(\theta_1) + l_2\cos(\theta_1+\theta_2) - 4.3)^2 - (l_1\sin(\theta_1) + l_2\sin(\theta_1+\theta_2) + 3.0)^2 + 2 \leq 0. \quad (38)$$

Let us consider one case where the initial joint pose is fixed as $\boldsymbol{\theta}(0) = [-1.521, 1.951, 1.353]^T$. The resulting globally optimal and periodic joint motions are shown without and with inequality constraints in Fig.1 and in Fig.2, respectively. Fig.2 shows that the inequality constraints are well satisfied.

When the initial joint pose is not fixed, it is observed that the proposed method using the second stage algorithm finds the best initial joint pose and the corresponding joint trajectory with globally minimmum performance cost. The simulation result for this example is not included owing to the space limit.

6 Conclusion

This paper presented a new method to globally resolve kinematic redundancy by using constrained optimization techniques such as Pontryagin's minimum principle and penalty function methods. These methods formulated the constrained optimization of the redundant manipulator as a kind of optimal redundancy control problem with inequality constraints. To incorporate joint limits, obstacles and joint velocity limits into the formulations, state and control variable inequality constraints were introduced, respectively.

The derived $(2n+2)$ necessary and periodic boundary conditions constitute a TPBVP. To solve this TPBVP, a two stage algorithm was proposed, in which given initial joint pose, optimal cost is obtained by using the steepest descent algorithm, and the resulting joint trajectory is globally optimal and periodic. If the initial joint pose is not fixed, the problem is transformed to find a global minimum of the performance cost which is a function of an $(n-m)$ dimensional parameter vector of the self-motion manifold subject to the initial end-effector location. This problem can be solved by some numerical search methods, but most of cost functions tend to be so complicated that it may be difficult to find a global minimum if *dor* is more than one. As a good candidate for this purpose, the evolutionary programming scheme may be considered.

To verify the effectiveness of the proposed method, a numerical example was shown by using a 3-*dof* planar manipulator with joint limits, joint velocity limits, and obstacles in the task space.

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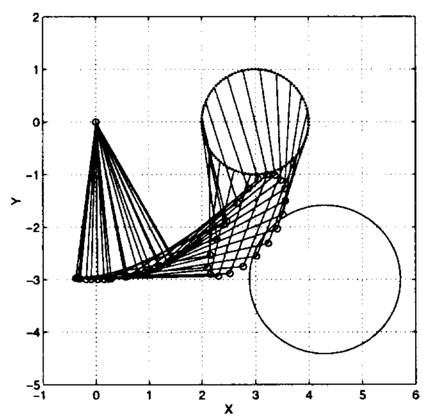

Figure 1: Globally optimal joint motion of 3-*dof* manipulator with the given initial pose where inequality constraints are not considered.

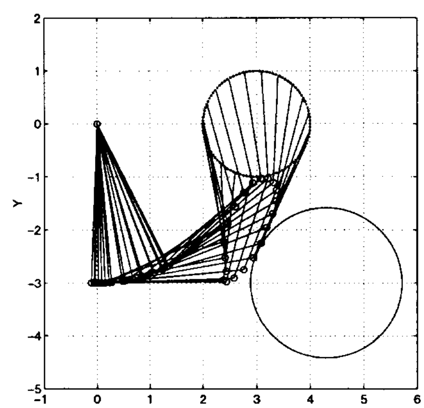

Figure 2: Globally optimal joint motion of 3-*dof* manipulator with the given initial pose where inequality constraints are considered.