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DISCRETE MODEL REDUCTION OVER DISC-TYPE ANALYTIC DOMAINS AND ∞-NORM ERROR BOUND

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Absrtact In this note, we propose the discrete model reduction method over disc-type analytic domains. We define Hankel singular value over the disc that is mapped by standard bilinear mapping. And GSPA(generalized singular perturbation approximation) and DT(direct truncation) are generalized to GSPA and DT over a disc. Furthermore we show that the reduced order model over a smaller domain has a smaller L_∞ norm error bound.

Keywords model reduction, analytic domain, Hankel singular value

1. INTRODUCTION

Discrete balanced model reductions such as GSPA[1]-[3] at $z=\pm 1$ and DT[4] are now standard results of the discrete model reduction theory. The errors in these methods are bounded in the sense of the L_∞ norm over the UCO(unit circle outside). Many researchers[2],[3],[5],[7] have found the L_∞ norm error bounds over the UCO and tried to reduce them. And the model reduction techniques considering frequency band are reported in many papers[3],[7]-[9]. We here consider a discrete model reduction over a disc-type analytic domain that is different from the outside of the unit circle. In the case of a continuous time system, Jonckheere *et al.*[8] proposed the model reduction method over a disc using a bilinear mapping that maps the right half plane onto the disc.

In this paper, we shall focus on disc domains that are bounded by a circle in the wide sense(*i.e.*, a straight line is also considered to be a circle in the complex plane and the disc domain could be the inside or the outside of the circle). Firstly, we define the standard bilinear mapping to the disc domain D and Hankel singular value over the disc. Secondly, we show the monotone property of Hankel singular values over the disc. As a results, when standard discrete balanced model reduction such as GSPA at at $z=\pm 1$ and DT is generalized to discrete balanced model reduction over a disc, we show that the reduced order model over a smaller disc domain has a smaller L_∞ norm error bound. By proposed method, we consider the working signal of system in a certain class of signals with limited frequency and transient speed. Finally, we give a small example to illustrate the validity of the proposed reduction procedure.

2. STANDARD BILINEAR MAPPING TO DISC TYPE ANALYTIC DOMAINS

Let us first have some definitions and consider the standard bilinear mapping that maps UCO onto a specific disc D.

Definition 1: A bilinear mapping from the complex plane to the complex plane is of the form

$$h(z)=\frac{\alpha z+\beta}{\gamma z+\delta} \tag{1}$$

and any circle in the complex plane can be mapped a circle in the complex plane by the bilinear mapping.

Of special interest is the mapping that maps UCO onto a disc D-a domain bounded by a circle. For the convenience of the exposition, we shall always assume that D is symmetric *w.r.t.* the real axis, and we want any two points symmetric about the real axis to be mapped to two points *w.r.t.* the real axis..

Definition 2: A bilinear mapping that maps UCO onto the disc D is called standard if it is of one of following forms:

$$h(z)=\frac{(a+b)z+(a-b)}{2z},\quad h(z)=\frac{(a-b)z+(a+b)}{2},\quad h(z)=\frac{(a+1)z+(a-1)}{z+1},\quad h(z)=\frac{(a-1)z+(a+1)}{z+1}, \tag{2}$$

where a, b are real numbers, and two intersecting points of the boundary of the circular disc and real axis are $(a, j0)$ and $(b, j0)$. And a intersecting point of the straight line and real axis is $(a, j0)$.

The generalized notion of disc D is represented in Fig. 1. Above standard mapping that maps UCO onto D preserves the symmetry of any two points *w.r.t.* the real axis. Let the poles of $G(z)$ be outside the disc D, and $h(z)$ be a standard bilinear mapping that maps UCO onto D. Then we know that $G(h(z))$ is asymptotically stable. In what follows, we shall be using standard bilinear mappings only.

Consequently, we know that $P \geq P_h$ from (13). If $r < 1$, there exist P and P_h satisfying $P_h - P \geq 0$ because $\Phi \leq 0$. And we can similarly prove for the observability grammians in (10) and (12). Therefore we conclude that $\sigma_i(G(z)) \geq \sigma_i(G(rz+r-1))$ ■

In theorem 1, a mapped analytic domain by $h(z) = rz + r - 1$ is represented in (a) of Fig. 2, where r is the radius of large circle. Note that $\sigma_i(G(h(z))) = \sigma_{Di}(G(z))$ and the large circle has to contain all poles of $G(z)$.

Theorem 2: Assume that $G(z) = (A, B, C, D)$ is bounded and analytic in UCO, then $\sigma_i(G(z)) \geq \sigma_i(G(\frac{(a-1)z+a+1}{2z}))$, where a is a real value and $a < -1$.

Proof: We here prove theorem 2 using similar procedures to the proof of theorem 1. From lemma 1, when a standard bilinear mapping is given by $h(z) = \frac{(a-1)z+a+1}{2z}$, a minimal realization of $G(h(z))$, (A_h, B_h, C_h), is obtained as

$$\begin{aligned} A_h &= -p(2A + pI + 2I)^{-1}, \\ B_h &= -\sqrt{2p}(2A + pI + 2I)^{-1}B, \\ C_h &= -\sqrt{2p}C(2A + pI + 2I)^{-1}, \end{aligned} \tag{16}$$

where $p = -(a+1) > 0$. Then we have two Lyapunov equations:

$$A_h P_h A_h^T - P_h + B_h B_h^T = 0, \tag{17}$$

$$A_h^T Q_h A_h - Q_h + C_h^T C_h = 0, \tag{18}$$

where P_h and Q_h are controllability and observability grammian of $G(\frac{(a-1)z+a+1}{2z})$, respectively. From (17), we obtain

$$AP_h A^T + \frac{p+2}{2} AP_h + \frac{p+2}{2} P_h A^T + (p+1)P_h - \frac{p}{2} BB^T = 0. \tag{19}$$

Subtracting (19) from (9), we have

$$A(P - P_h)A^T - (P - P_h) + \Phi = 0, \tag{20}$$

$$\Phi = -\frac{p+2}{2} AP_h - \frac{p+2}{2} P_h A^T - (p+2)P_h + \frac{p+2}{2} BB^T, \tag{21}$$

and if $\Phi \geq 0$, there exist P and P_h satisfying $P - P_h \geq 0$. From (19), we find BB^T and substitute it into (21). Then we have a simplified form:

$$\Phi = (A + I)\eta_2 P_h (A + I)^T \geq 0, \quad \eta_2 = \frac{p+2}{p}. \tag{22}$$

And we can similarly prove for the observability grammians in (10) and (18). Therefore we conclude that $\sigma_i(G(z)) \geq \sigma_i(G(\frac{(a-1)z+a+1}{2z}))$. ■

A mapped analytic domain by $h(z) = \frac{(a-1)z+a+1}{2z}$ is represented in (b) of Fig. 2.

Lemma 3: Let D_s and D_l be the two discs symmetric *w.r.t.* the real axis. $G(z)$ is bounded and analytic in D_l and $D_s \subseteq D_l$. The boundaries of D_s and D_l have a common point on the real axis. Then

$$\sigma_{D_s i}(G(z)) \leq \sigma_{D_l i}(G(z)), \tag{23}$$

where the subscript 's' is used to indicate the small domain, and 'l' the large domain.

Proof: We construct a bilinear mapping $h(z)$ such that D_l is mapped onto UCO. Since $D_s \subseteq D_l$, D_s is mapped onto a subset of UCO. If we choose $h(z)$ such that the common point of the two circles is mapped to the point at -1 of the complex plane, the boundary of D_s is also mapped to a circle outside the unit circle with the point at -1. That is to say, there exists a standard bilinear mapping $h(\cdot)$ such that

$h(\text{UCO}) = D_l$ and $h(r\text{UCO} + r - 1) = D_s$, otherwise
$h(\text{UCO}) = D_l$ and $h(\frac{(a-1)\text{UCO}+a+1}{2\text{UCO}}) = D_s$,

where $r (> 1)$ and $a (< -1)$ are real values. We have

$$\begin{aligned} &\sigma_{D_s i}(G(z)) = \sigma_i(G(h(rz + r - 1))) = \sigma_i(F(rz + r - 1)), \text{ or} \\ &\sigma_{D_s i}(G(z)) = \sigma_i(G(h(\tfrac{(a-1)z+a+1}{2z}))) \\ &\qquad = \sigma_i(F(\tfrac{(a-1)z+a+1}{2z})), \end{aligned} \tag{24}$$

where $\sigma_{D_l i}(G(z)) = \sigma_i(G(h(z))) = \sigma_i(F(z))$. For the case of the disc where is in the right or left hand side of the straight line, we see easily that two discs satisfying $D_s \subseteq D_l$ have a common point at infinity. If we choose $h(z)$ such that the common point is mapped to the point at -1 of the complex plane, then we can get the same result as the case of the disc with circular bound. Consequently, we see that $\sigma_{D_s i}(G(z)) \leq \sigma_{D_l i}(G(z))$ from theorem 1 and 2. ■

Theorem 3: Let D_s and D_l be the two discs symmetric *w.r.t.* the real axis and $D_s \subseteq D_l$. $G(z)$ is bounded and analytic in D_l. Then

$$\sigma_{D_s i}(G(z)) \leq \sigma_{D_l i}(G(z)). \tag{25}$$

3. HANKEL SINGULAR VALUE OVER DISC D IN DISCRETE TIME SYSTEM

In this section, the Hankel singular value over D in discrete time system is defined. There are two standard bilinear mappings that map UCO onto a given disc D, and we prove that the Hankel singular values over the disc are independent of the choice of the standard bilinear mappings.

Definition 3: Let $G(z)$ be a transfer function matrix with McMillan degree n that is bounded and analytic in D. $h(z)$ is a standard bilinear mapping, *i.e.*, $h(\mathrm{UCO})=D$. The ith Hankel singular value over D is defined as

$$\sigma_{Di} := \sigma_i(G(h(z))), \tag{3}$$

where $\sigma_i(F(z))$ is ith standard Hankel singular value of $F(z)$. And $\|G(z)\|_{HD} := \sigma_1(G(h(z)))$ is the Hankel norm of $G(z)$ over D, where $\sigma_1(F(z))$ is the largest Hankel singular value of $F(z)$.

Lemma 1 [9]: Let $G(z)$ be the transfer function matrix same as in definition 3. $h(z)=(\alpha z+\beta)/(\gamma z+\delta)$ is a bilinear mapping with $w=|\alpha\delta-\beta\gamma|^{1/2}>0$. If (A,B,C,D) is a minimal realization of $G(z)$, then a minimal realization of $G(h(z))$, (A_h, B_h, C_h, D_h) is given by

$$\begin{aligned} A_h &= (\alpha I-\gamma A)^{-1}(\delta A-\beta I), \\ B_h &= w(\alpha I-\gamma A)^{-1}B, \\ C_h &= wC(\alpha I-\gamma A)^{-1}, \\ D_h &= D+wC(\alpha I-\gamma A)^{-1}B. \end{aligned} \tag{4}$$

As already mentioned, we show that the Hankel singular values over D are independent of the choice of the standard bilinear mappings that map UCO onto a disc.

Lemma 2: Let $h_1(z)$ and $h_2(z)$ be the two standard mappings that map UCO onto a disc D, and $G(z)$ be a transfer function matrix bounded and analytic in D. We then have

$$\sigma_i(G(h_1(z)))=\sigma_i(G(h_2(z))). \tag{5}$$

Proof: We shall prove the lemma only for the case where the disc D is the outside of the circle. For the case where D is the inside of the circle, the proof is similar. Two standard mappings to the outside of the circle are given by

$$h_1(z)=\frac{(a-b)z+(a+b)}{2},\quad h_2(z)=\frac{(b-a)z+(b+a)}{2}. \tag{6}$$

From lemma 1, we can get minimal realizations of $G(h_1(z))$ and $G(h_2(z))$. And we have the following Lyapunov equations:

$$\begin{aligned} (a-b)^{-2}\{2A &-(a+b)I\}P_1\{2A-(a+b)I\}^T \\ &-P_1+w^2(a-b)^{-2}BB^T=0, \end{aligned} \tag{7}$$

$$\begin{aligned} (b-a)^{-2}\{2A &-(b+a)I\}P_2\{2A-(b+a)I\}^T \\ &-P_2+w^2(b-a)^{-2}BB^T=0, \end{aligned} \tag{8}$$

where $w=|2(a-b)|^{1/2}$, and from (7) and (8) we see that $P_1=P_2$. And we can similarly prove that the observability grammians of the $G(h_1(z))$ and $G(h_2(z))$ are the same. Therefore we conclude that the Hankel singular values of $G(h_1(z))$ and $G(h_2(z))$ are the same. ■.

4. MONOTONE PROPERTY OF HANKEL SINGULAR VALUE OVER DISC D

The tradeoff between the size of the disc domain and model reduction error is an important issue. And we show that the discrete system analytic over a smaller domain has a smaller Hankel singular value.

Theorem 1: Assume that $G(z)=(A,B,C,D)$ is bounded and analytic in UCO, then $\sigma_i(G(z))\geq\sigma_i(G(rz+r-1))$, where r is a real value and $r\geq 1$.

Proof: We consider the following standard Lyapunov equations:

$$APA^T-P+BB^T=0, \tag{9}$$

$$A^TQA-Q+C^TC=0. \tag{10}$$

From lemma 1, when a standard bilinear mapping is given by $h(z)=rz+r-1$, we have

$$r^{-2}(A-rI+I)P_h(A-rI+I)^T-P_h+r^{-1}BB^T=0, \tag{11}$$

$$r^{-2}(A-rI+I)^TQ_h(A-rI+I)-Q_h+r^{-1}C^TC=0, \tag{12}$$

where P_h and Q_h are controllability and observability grammian of $G(rz+r-1)$, respectively. Subtracting (11) from (9), we obtain

$$A(P-P_h)A^T-(P-P_h)+\Phi=0, \tag{13}$$

$$\begin{aligned} \Phi=(r-1)AP_h+(r-1)P_hA^T-P_h \\ -(1-2r)P_h+(1-r)BB^T, \end{aligned} \tag{14}$$

and if $\Phi\geq 0$, there exist P and P_h satisfying $P-P_h\geq 0$ because all the eigenvalues of A are inside the unit circle. From (11), we find $(1-r)BB^T$ and substitute it into (14). Then since $P_h\geq 0$ and $r>1$, we have

$$\begin{aligned} \Phi &= (r-1)\{AP_h+P_hA^T\}-P_h-(1-2r)P_h-r^{-1}AP_hA^T \\ &\quad +AP_hA^T+(r-1)r^{-1}AP_h+(r-1)r^{-1}P_hA^T \\ &\quad -(r-1)\{AP_h+P_hA^T\}-(1-2r)r^{-1}P_h+(1-2r)P_h \\ &= (1-r^{-1})\{AP_hA^T+AP_h+P_hA^T+P_h\} \\ &= (A+I)\eta_1P_h(A+I)^T\geq 0,\qquad \eta_1=\frac{r-1}{r}. \end{aligned} \tag{15}$$

Proof: Since $D_s \subseteq D_l$, there exists a disc D_m symmetric *w.r.t.* the real axis that satisfies $D_s \subseteq D_m \subseteq D_l$, and the boundary of D_m has a common point on the real axis with both of two discs as seen in Fig. 3. Then from lemma 3, we have

$$\sigma_{D_s i}(G(z)) \leq \sigma_{D_m i}(G(z)) \leq \sigma_{D_l i}(G(z)), \qquad (26)$$

where $\|G(z)\|_{HD_s} \leq \|G(z)\|_{HD_l}$ ■

5. DISCRETE MODEL REDUCTION AND L_∞ NORM ERROR BOUND

We propose the discrete balanced model reduction over disc D. For the given reduction technique, we can choose an analytic disc domain to reduce L_∞ norm error bound. And we apply this concept of discrete model reduction over D to the discrete balanced model reduction such as GSPA[2] of balanced system at $z = \pm 1$ and DT[4].

Theorem 4: Let D be a disc symmetric *w.r.t.* the real axis. $G(z)$ is bounded and analytic in D. Then the reduced order model by the discrete balanced model reduction over D, $G_b(z)$, is also analytic in D.

Proof: Let $h(z)$ be a standard bilinear mapping that maps UCO onto D, and $G_b(h(z))$ be the reduced order model by the standard discrete balanced model reduction of $G(h(z))$. Since $G(z)$ is analytic in D, $G(h(z))$ must be analytic in UCO, and so is $G_b(h(z))$, which means that $G_b(z)$ is analytic in D. ■

Theorem 5: Let D be a disc symmetric *w.r.t.* the real axis. $G(z)$ is bounded and analytic in D. The Hankel singular values of $G(z)$ over D are given by

$$\sigma_{D1} > \sigma_{D2} > \sigma_{D3} > \cdots > \sigma_{Dn} > 0, \qquad (27)$$

where σ_{Di} has muliplicity r_i. Then an error bound between $G(z)$ and $G_b(z)$ is as follows:

$$\sup_{z \in D} \sigma_{\max}(G(z) - G_b(z)) \leq 2(\sigma_{D(k+1)} + \cdots + \sigma_{Dn}), \qquad (28)$$

where $(r_1 + \cdots + r_k)$ is the reduced order and $\sigma_{\max}$ is the largest singular value.

Proof: Let $h(z)$ be a standard bilinear mapping that maps UCO onto D. From the standard discrete balanced model reduction theory[1]-[5], we have

$$\sup_{\hat{z} \in \mathrm{UCO}} \sigma_{\max}(G(h(\hat{z})) - G_b(h(\hat{z}))) \leq 2(\sigma_{D(k+1)} + \cdots + \sigma_{Dn}),$$

when we assume that $z = h(\hat{z})$, it is easily seen that $\hat{z} \in \mathrm{UCO} \Leftrightarrow z \in D$. Therefore (28) is satisfied. ■

From theorem 3, we know that if $D_s \subseteq D_l$ then the corresponding L_∞ norm error bound is represented by

$$E_l = 2(\sigma_{D_l(k+1)} + \cdots + \sigma_{D_l n}) \geq E_s = 2(\sigma_{D_s(k+1)} + \cdots + \sigma_{D_s n}).$$

This means that the reduced order model over a smaller domain has a smaller model reduction error.

Remark: For an unstable discrete systems, if we choose a disc such that the circle contains all the poles of the unstable system, it is able to reduce the unstable discrete system over the disc[8].

Algorithm and Example: Algorithm is as follows:

step 1 Find a standard bilinear mapping $h(z)$ for a given disc D.
step 2 Compute $F(z) = G(h(z))$ using lemma 1.
step 3 Find a standard balanced reduced order model of $F(z)$.
step 4 Construct the inverse bilinear mapping of $h(z)$, $h^{-1}(z)$. Then the reduced order model over D is given by $G_r(z) = F_r(h^{-1}(z))$, where $F_r(z)$ is a reduced order model of $F(z)$.

Example: We consider a 7th-order discrete transfer function to show the validity of the proposed the discrete model reduction procedure.

$$G(z) = \frac{0.1551z^7 + 0.3705z^6 + 0.3830z^5 + 0.1886z^4 - 0.0812z^3 - 0.2756z^2 - 0.2631z - 0.0995}{z^7 + 3.0954z^6 + 4.8744z^5 + 4.8110z^4 + 3.0636z^3 + 1.2214z^2 + 0.2325z + 0.0081}.$$

It is reduced to 4th-order system by GSPA at $z = \pm 1$ or DT. Then we compare the standard GSPA over UCO with the proposed GSPA over D with its intersecting points, (-5, j0) and (1, j0). The frequency responses are shown in Fig. 4 and 5. We have frequency responses in **w**-domain with sampling time T = 10[ms]. Therefore the area with frequency above Nyquist frequency is neglected. From Fig. 5, we see that GSPA and DT over the disc D have good approximations in low frequency area.

The Hankel singular value over UCO is given by (0.543126, 0.491837, 0.140245, 0.120304, 0.096332, 0.068325, 0.064708) and L_∞ norm bound over UCO is 0.458733. The Hankel singular value over D is given by (0.075382, 0.007811, 0.004818, 0.001287, 0.000015, 0.0000004, 0.00000002), and L_∞ norm bound over D is 0.00003169.

6. CONCLUSION

We defined the Hankel singular value over the disc that is mapped by a standard bilinear mapping. And we generalized the standard GSPA at $z = \pm 1$ and DT to the GSPA at $z = \pm 1$ and DT over a specific disc domain. We showed that the reduced order model over a smaller disc domain has a smaller L_∞ norm error bound. We gave a small example to illustrate the validity of the proposed reduction procedure.

REFERENCES

[1] K. V. Fernando and H. Nicholson, "Singular per-turbational approximations for discrete-time balanced systems," *IEEE Trans. Automat. Contr.*, vol. 28, no. 2, pp. 240-242, 1983.

[2] Y. Liu and B. D. O. Anderson, "Singular per-turbation approximation of balanced systems," *Int. J. Contr.*, vol. 50, no. 4, pp. 1379-1405, 1989.

[3] U. M. AL-Saggaf and G. F. Franklin, "Model reduction via balanced realizations: An extention and frequency weighting techniques," *IEEE Trans. Automat. Contr.*, vol. 33, no. 7, pp. 687-692, 1988.

[4] I. Pernebo and L. M. Silverman, "Model reduction via balanced state space representations," *IEEE Trans. Automat. Contr.*, vol. 27, no. 2, pp. 382-387, 1982.

[5] U. M. AL-Saggaf and G. F. Franklin, "An error bound for a discrete reduced order model of a linear multivariable system," *IEEE Trans. Automat. Contr.*, vol. 32, no. 9, pp. 815-819, 1987.

[6] D. Hinrichsen and A. J. Pritchard, "An improved error estimate for reduced-order models of discrete-time systems," *IEEE Trans. Automat. Contr.*, vol. 27, no. 2, pp. 382-387, 1982.

[7] Enns, "Model reduction for control system design," Ph.D. Dissertation, Department of Aeronautics and Astronautics, Stanford University, 1984.

[8] E. Jonckheere and R. Li, "Generalization of optimal Hankel-norm and balanced model reduction by bilinear mapping," *Int. J. Contr.*, vol. 45, no. 5, pp. 1751-1769, 1987.

[9] K. Zhou, "Error bounds for frequency weighted balanced truncation and relative error model reduction," in *Proc. 32nd Conf. Decis. Contr.*, San Antonio, TX, pp. 3347-3352, Dec. 1993.

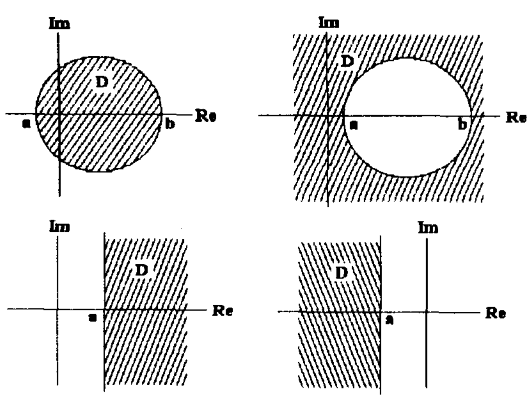
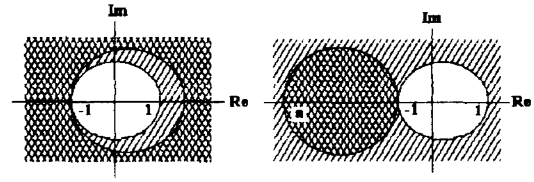


Fig. 1. The generalized notion of a dosc D.



(a) $h(z) = rz + r - 1$ (b) $h(z) = \dfrac{(a-1)z + a + 1}{2z}$

Fig 2. The analytic domains for theorem 1 and 2.

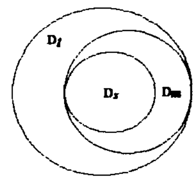


Fig. 3. The disc D_m in the proof of theorem 3.

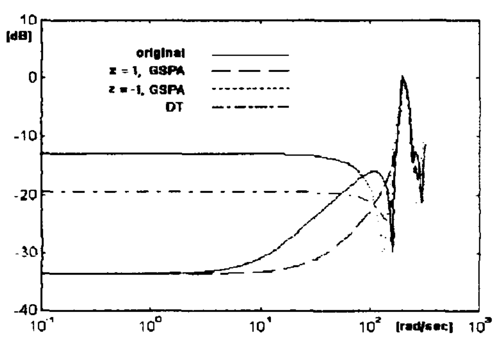


Fig. 4. Reduced order models over UCO.

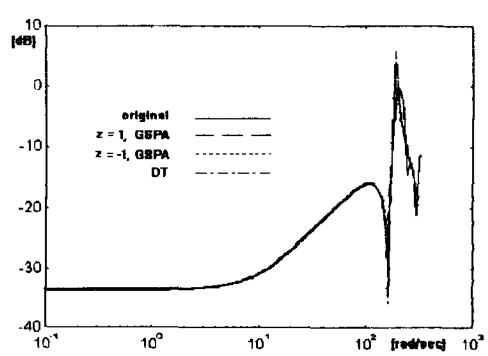


Fig. 5. Reduced order models over the disc D.