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An Analysis on Worst-case State Estimation in Standard $\mathcal{H}_\infty$ State-Space Solution

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Abstracts Worst-case state estimation will be proposed in this paper. By using the worst-case disturbance and worst-case state estimation, we can obtain right/left constrained coprime factors. If constrained coprime factors are used in designing a controller, the infinity-norm of closed-loop transfer matrix can be smaller than any constant $\gamma(>\gamma_{opt})$ without matrix dilation optimization. The derivation of left/right constrained coprime factors is achieved by doubly coprime factorization for the plant constrained by the infinity norm. And the parameterization of stabilizing controllers gives us easily understanding for $\mathcal{H}_\infty$ control theory.

Keywords $\mathcal{H}_\infty$ control, Doubly Coprime Factorization

1. INTRODUCTION

Doubly coprime factorization for linear time invariant systems is used to parameterize all stabilizing controllers. The parameterization of stabilizing controllers was first introduced by Youla *et al*[5]. Especially, the Youla parameterization provides a systematic way to choose the (optimal) stabilizing controllers[2]. The conventional Youla parameterization gives the structure of $\mathcal{H}_2$ controller and free Youla $\boldsymbol{Q}$ parameter. In conventional Youla parameterization approach to $\mathcal{H}_\infty$ control, the $\boldsymbol{Q}$ design to minimizing ∞-norm of the closed-loop transfer matrix should be achieved by matrix dilation optimization.

2. PROBLEM STATEMENT

Youla parameterization for original plant gives the standard $\mathcal{H}_2$ controller structure and free parameter $\boldsymbol{Q}$. In conventional Youla parameterization approach to $\mathcal{H}_\infty$ control, the parameter $\boldsymbol{Q}$ has been designed by using matrix dilation optimization according to the infinity norm constraint for closed-loop transfer matrix. In this case, since the state dimension of $\boldsymbol{Q}$ is $2n$, the apparent dimension of controller is $3n$, finally, tedious manipulations will produce cancelations resulting in the n dimensional controller[4].

To overcome these problems, firstly, we define the worst-case state estimation, secondary, we suggest constrained right/left coprime factors which can give a controller which can satisfy the infinity norm constraint for the closed-loop transfer matrix.

3. UNCONSTRAINED/CONSTRAINED COPRIME FACTORS

For a original plant $\boldsymbol{G}(s)=\boldsymbol{C}_2(s\boldsymbol{I}-\boldsymbol{A})^{-1}\boldsymbol{B}_2+\boldsymbol{D}_{22}$, the generalized plant can be expressed by

$$\begin{aligned}\dot{\boldsymbol{x}}(t) &= \boldsymbol{A}\boldsymbol{x}(t)+\boldsymbol{B}_1\boldsymbol{w}(t)+\boldsymbol{B}_2\boldsymbol{u}(t)\\ \boldsymbol{z}(t) &= \boldsymbol{C}_1\boldsymbol{x}(t)+\boldsymbol{D}_{11}\boldsymbol{w}(t)+\boldsymbol{D}_{12}\boldsymbol{u}(t)\\ \boldsymbol{y}(t) &= \boldsymbol{C}_2\boldsymbol{x}(t)+\boldsymbol{D}_{21}\boldsymbol{w}(t)+\boldsymbol{D}_{22}\boldsymbol{u}(t)\end{aligned} \quad (1)$$

where $\boldsymbol{x}(t)$ is state vector ,$\boldsymbol{u}(t)$ is control input vector, $\boldsymbol{z}(t)$ is controlled output vector, $\boldsymbol{y}(t)$ is output vector and $\boldsymbol{w}(t)$ is the disturbance which is caused by unmodeled dynamics, modeling error and etc. We may omit time t for any vectors.

Assume that the following properties are satisfied for a given system of Eq.(1) to describe controller as simple as possible:

1. $(\boldsymbol{A},\boldsymbol{B}_1,\boldsymbol{C}_1)$ is stabilizable and detectable.
2. $(\boldsymbol{A},\boldsymbol{B}_2,\boldsymbol{C}_2)$ is stabilizable and detectable.
3. $\boldsymbol{D}_{12}^T[\boldsymbol{C}_1\ \ \boldsymbol{D}_{12}]=[\boldsymbol{0}\ \ \boldsymbol{I}]$.
4. $\begin{bmatrix}\boldsymbol{B}_1\\ \boldsymbol{D}_{21}\end{bmatrix}\boldsymbol{D}_{21}^T=\begin{bmatrix}\boldsymbol{0}\\ \boldsymbol{I}\end{bmatrix}$.
5. $\boldsymbol{D}_{11}=\boldsymbol{0}$ and $\boldsymbol{D}_{22}=\boldsymbol{0}$.

where $\boldsymbol{0}$ is zero matrix of suitable dimension. If the above five assumptions are not satisfied for the generalized plant, the loop transformation and loop scaling[1] may be needed case by case to satisfy these assumptions. We utilize following equations for generalized plant satisfying the above assumptions without derivation and proof because these can be found in some references[1, 4, 6]:

$$\boldsymbol{X}_\infty = Ric\begin{bmatrix}\boldsymbol{A} & \gamma^{-2}\boldsymbol{B}_1\boldsymbol{B}_1^T-\boldsymbol{B}_2\boldsymbol{B}_2{}^T\\ -\tilde{\boldsymbol{C}}_1^T\tilde{\boldsymbol{C}}_1 & -\boldsymbol{A}^T\end{bmatrix} \quad (2)$$

$$\boldsymbol{Y}_\infty = Ric\begin{bmatrix}\boldsymbol{A}^T & \gamma^{-2}\boldsymbol{C}_1^T\boldsymbol{C}_1-\boldsymbol{C}_2{}^T\boldsymbol{C}_2\\ -\tilde{\boldsymbol{B}}_1\tilde{\boldsymbol{B}}_1^T & -\boldsymbol{A}\end{bmatrix} \quad (3)$$

$$\begin{aligned}\tilde{\boldsymbol{B}}_1 &= \boldsymbol{B}_1(\boldsymbol{I}-\boldsymbol{D}_{21}^T\boldsymbol{D}_{21})\\ \tilde{\boldsymbol{C}}_1 &= (\boldsymbol{I}-\boldsymbol{D}_{12}\boldsymbol{D}_{12}^T)\boldsymbol{C}_1\\ \boldsymbol{F}_\infty &= -\boldsymbol{B}_2^T\boldsymbol{X}_\infty\\ \boldsymbol{H}_\infty &= -\boldsymbol{Y}_\infty\boldsymbol{C}_2^T\\ \boldsymbol{Z}_\infty &= (\boldsymbol{I}-\gamma^{-2}\boldsymbol{Y}_\infty\boldsymbol{X}_\infty)^{-1}\end{aligned}$$

where $\boldsymbol{X}_\infty\geq 0$,$\boldsymbol{Y}_\infty\geq 0$, $\rho(\boldsymbol{X}_\infty\boldsymbol{Y}_\infty)<\gamma^2$, and the Riccati solutions $\boldsymbol{X}_\infty$ and $\boldsymbol{Y}_\infty$ exist(that is, Hamiltonian matrices have no imaginary eigenvalues). The solution of Riccati equation is a real, symmetric and semi-positive definite matrix. When $\gamma\to\infty$, we define $\boldsymbol{X}_2:=\boldsymbol{X}_\infty$,$\boldsymbol{Y}_2:=\boldsymbol{Y}_\infty$ and matrix $\boldsymbol{Z}_\infty$ is equal to identity one. Also, we define $\boldsymbol{F}_2:=-\boldsymbol{B}_2^T\boldsymbol{X}_2$ and $\boldsymbol{H}_2:=-\boldsymbol{Y}_2\boldsymbol{C}_2^T$.

We can also obtain well-known results for doubly coprime factorization technique and conventional Youla parameterization of stabilizing controllers in some reference[1, 6]. Let's the subscript "ur" represent "unconstrained right coprime factor" and "ul" "unconstrained left coprime factor".

Remark 1 *For a given original plant:*

$$\boldsymbol{G}(s)=\left[\begin{array}{c|c}\boldsymbol{A} & \boldsymbol{B}_2\\ \hline \boldsymbol{C}_2 & \boldsymbol{0}\end{array}\right]=\boldsymbol{C}_2(s\boldsymbol{I}-\boldsymbol{A})^{-1}\boldsymbol{B}_2. \quad (4)$$

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The doubly coprime factorization is given by

$$\begin{bmatrix} Y_{ur} & X_{ur} \\ -N_{ul} & M_{ul} \end{bmatrix}\begin{bmatrix} M_{ur} & -X_{ul} \\ N_{ur} & Y_{ul} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \quad (5)$$

with each transfer functions in $\mathcal{RH}_\infty$, and where

$$\begin{bmatrix} Y_{ur} & X_{ur} \\ -N_{ul} & M_{ul} \end{bmatrix} = \left[\begin{array}{c|cc} A+H_2C_2 & B_2 & -H_2 \\ \hline -F_2 & I & 0 \\ -C_2 & 0 & I \end{array}\right] \quad (6)$$

and

$$\begin{bmatrix} M_{ur} & -X_{ul} \\ N_{ur} & Y_{ul} \end{bmatrix} = \left[\begin{array}{c|cc} A+B_2F_2 & B_2 & -H_2 \\ \hline F_2 & I & 0 \\ C_2 & 0 & I \end{array}\right]. \quad (7)$$

These are elementary results for doubly coprime factorization.

Above results can be independently derived from internal stability theory. Also, coprime factors are normalized such as $\|[N_{ul} \ \ M_{ul}]\|_\infty = 1$ for all freqencies.

The closed-loop transfer matrix can be expressed by

$$\begin{aligned} \mathcal{F}_l(P,K) &= P_{11} + P_{12}(M_{ur}Q - X_{ul})M_{ul}P_{21} \\ &= (P_{11} - P_{12}X_{ul}M_{ul}P_{21}) \\ &\quad +(P_{12}M_{ur})Q(M_{ul}P_{21}) \\ &= T_{11} + T_{12}QT_{21} \end{aligned} \quad (8)$$

in which

$$\begin{bmatrix} T_{11} & T_{12} \\ T_{21} & 0 \end{bmatrix} = \left[\begin{array}{cc|cc} A+B_2F_2 & -H_2C_2 & -H_2D_{21} & B_2 \\ 0 & A+H_2C_2 & B_1+H_2D_{21} & 0 \\ \hline C_1+D_{12}F_2 & C_1 & 0 & D_{12} \\ 0 & C_2 & D_{21} & 0 \end{array}\right].$$

Youla parameterization approach to $\mathcal{H}_\infty$ control is minimizing ∞-norm of Eq.(8). This replaces the design problem selecting a stabilizing controller with that of selecting a suitable Q such that

$$\inf_K \|\mathcal{F}_l(P,K)\|_\infty = \inf_Q \|T_{11} + T_{12}QT_{21}\|_\infty \leq \gamma. \quad (9)$$

Matrix dilation optimization should be used to satisfy the infinity norm constraint of closed-loop transfer matrix, in other words, Youla Q_{opt} parameter should be designed according to the infinity norm constraint. Since state dimension of Q_{opt} is $2n$, dimension of controller becomes $3n$. However, tedious manipulations bring about cancelations resulting in the n-dimensional controller.

3.1. Plant Constrained by the Infinity Norm

Definition 1 *Consider the system of Eq.(1) satisfying above five assumptions. Inequality constraint defined for any positive constant $\gamma > \gamma_{opt}$ as*

$$\|z\|_2^2 \leq \gamma^2\|w\|_2^2 \quad (10)$$

is said to be infinity norm constraint, and the plant induced from infinity norm constraint is said to be the plant constrained by the infinity norm.

This definition is very meaningful in view of the disturbance attenuation, and it gives us to the inequality constraint for ∞-norm of closed-loop transfer matrix as shown in Fig.1, namely, $\|\mathcal{F}_l(P,K)\|_\infty \leq \gamma$, where P is generalized plant and K is controller.

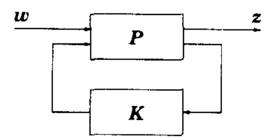

Fig. 1 Infinity-norm constraint

Proposition 1 *Consider a linear system of Eq.(1) for which assumptions 1-5 hold. Suppose the matrix X_∞ of Eq.(2) exists, then the closed loop dynamics is expressed by*

$$\dot{x} = (A + \frac{1}{\gamma^2}B_1B_1^TX_\infty - B_2B_2^TX_\infty)x \quad (11)$$

and the matrix $A + \frac{1}{\gamma^2}B_1B_1^TX_\infty - B_2B_2^TX_\infty$ has all its eigenvalues in the open left half of the complex plane. The worst case disturbance input is

$$w = \gamma^{-2}B_1^TX_\infty x \quad (12)$$

and the optimal control input is

$$u = F_\infty x = -B_2^TX_\infty x. \quad (13)$$

The closed-loop dynamics of Eq.(11) is the dynamics constrained by the infinity norm constraint of Eq.(10).

If the solution of Riccati equation X_∞ exists, then the closed-loop dynamics of Eq.(11) is asymptotically stable since the riccati operator of Eq.(2) gives the stable invariant subspace by properties of riccati operator. If we assume that the symmetric and semi-positive definite matrix X_∞ exists, we can differentiate Lyapunov candidate of $x^TX_\infty x$:

$$\begin{aligned} &\frac{d}{dt}(x^TX_\infty x) \\ = \ &-\|z\|_2^2 + \gamma^2\|w\|_2^2 \\ &+\|u + B_2^TX_\infty x\|_2^2 - \gamma^2\|w - \gamma^{-2}B_1^TX_\infty x\|_2^2 \end{aligned} \quad (14)$$

where the Riccati equation for X_∞ is

$$\begin{aligned} &A^TX_\infty + X_\infty A \\ &+C_1^TC_1 + \gamma^{-2}X_\infty B_1B_1^TX_\infty - X_\infty B_2B_2^TX_\infty = 0. \end{aligned}$$

Assume that $x(0) = x(\infty) = 0$ and integrate Eq.(14) from $t = 0$ to $t = \infty$, then we obtain:

$$\|z\|_2^2 - \gamma^2\|w\|_2^2 = \|u + B_2^TX_\infty x\|_2^2 - \gamma^2\|w - \gamma^{-2}B_1^TX_\infty x\|_2^2. \quad (15)$$

Hence, the infinity norm constraint of Eq.(10) can be interpreted as Eq.(15). Note that $w := \gamma^{-2}B_1^TX_\infty x$ is the worst case disturbance input in the sense that it maximizes the quantity $\|z\|_2^2 - \gamma^2\|w\|_2^2$ in Definition 1 for the minimizing value of $u = -B_2^TX_\infty x$; that is the u making $u + B_2^TX_\infty x = 0$ and w making $w - \gamma^{-2}B_1^TX_\infty x = 0$ are values satisfying a saddle point condition. Infinity norm constraint gives the functional relations of Eq.(12) and (13) between disturbance input, control input and state vector. Then, the closed-loop dynamics for generalized plant Eq.(1) is equal to Eq.(11) in Proposition 1.

Proposition 2 *Consider the closed-loop dynamics of Eq.(11) in Proposition 1. Suppose the X_∞, Y_∞ and Z_∞*

exist. Then the plant constrained by the infinity norm constraint of Eq.(10) in Definition 1 is expressed by

$$P_s = \left[\begin{array}{c|c} A + \frac{1}{\gamma^2} B_1 B_1^T X_\infty + \frac{1}{\gamma^2} Z_\infty Y_\infty X_\infty B_2 B_2^T X_\infty & Z_\infty B_2 \\ \hline C_2 & 0 \end{array}\right] \quad (16)$$

where P_s denotes the plant constrained by the infinity norm. The effect on system matrix caused by worst-case disturbance input is

$$\frac{1}{\gamma^2} B_1 B_1^T X_\infty \quad (17)$$

and the effect on system matrix caused by worst-case state estimation is

$$\frac{1}{\gamma^2} Z_\infty Y_\infty X_\infty B_2 B_2^T X_\infty. \quad (18)$$

The plant constrained by the infinity norm considers the effect which can be caused by worst-case disturbance and worst-case state estimation.

In the problem of full state feedback, since the measure of state vector is available, $Y_\infty = 0$ and $Z_\infty = I$. However, without loss of generality, the plant constrained by the infinity norm can be suggested from a simple algebraic calculation for Eq.(11):

$$\begin{aligned} \dot{x} &= (A + \gamma^{-2} B_1 B_1^T X_\infty - Z_\infty Z_\infty^{-1} B_2 B_2^T X_\infty) x \\ &= (A + \gamma^{-2} B_1 B_1^T X_\infty + \gamma^{-2} Z_\infty Y_\infty X_\infty B_2 B_2^T X_\infty) x \\ &\quad + Z_\infty B_2 u \end{aligned} \quad (19)$$

where Z_∞ is real symmetric positive definite matrix.

However, in the problem of output feedback, can we say the plant constrained by the infinity norm is expressed like Eq.(16)? The answer is "positive". That is the reason why there is no interference on the stability between observer and system dynamics by the separation principle. Therefore, let's the stability for the state error dynamics investigate. Firstly, if (C_2, A) is detectable, the observer for generalized plant of Eq.(1) can be given by

$$\begin{gathered} \dot{\hat{x}} = A\hat{x} + B_1 \hat{w} + B_2 u + Z_\infty H_\infty (C_2 \hat{x} - y) \quad (20) \\ u = F_\infty \hat{x} \quad \text{and} \quad \hat{w} = \gamma^{-2} B_1^T X_\infty \hat{x} \end{gathered}$$

where $Z_\infty H_\infty$ is any matrix such that $A + \gamma^{-2} B_1 B_1^T X_\infty + Z_\infty H_\infty$ is stable. Above observer is the typical *Luenberger* observer[6], and the observer of Eq.(20) is also that of $\mathcal{H}_\infty$ central controller. Define the state error vector as $e = x - \hat{x}$ and consider the state error dynamics to investigate the stability of state error dynamics which is obtained by subtracting Eq.(20) from Eq.(1):

$$\dot{e} = (A + \gamma^{-2} B_1 B_1^T X_\infty + Z_\infty H_\infty C_2) e. \quad (21)$$

Let's differentiate the Lyapunov candidate of $\frac{1}{2}\gamma^2 e^T Z^{-1} e$ for the state error dynamics

$$\begin{aligned} &\frac{d}{dt}\left(\frac{1}{2}\gamma^2 e^T Z^{-1} e\right) \\ &= -\frac{1}{2}\left(\gamma^2 \|B_1^T Z^{-1} e\|_2^2 + \gamma^2 \|C_2 e\|_2^2 + \|B_2^T X_\infty e\|_2^2\right) \le 0 \end{aligned} \quad (22)$$

where the Riccati equation for Z is

$$\begin{aligned} &Z(A + \gamma^{-2} B_1 B_1^T X_\infty)^T + (A + \gamma^{-2} B_1 B_1^T X_\infty) Z \\ &+ B_1 B_1^T - Z C_2^T C_2 Z + \gamma^{-2} Z X_\infty B_2 B_2^T X_\infty Z = 0, \end{aligned} \quad (23)$$

Since the stabilizability of (A, B_2) and detectability of (A, C_2) ensure that $\lim_{t \to \infty} e(t) = 0$ when $\frac{d}{dt}\frac{1}{2}\gamma^2 (e^T Z^{-1} e) = 0$, therefore, the state error dynamics of Eq.(21) is asymptotically stable.

The Riccati equation of Eq.(23) is obtained by the similarity transformation of Hamiltonian matrix associated with Y_∞ of Eq.(3). Introducing the transformation matrix

$$T = \begin{bmatrix} I & -\gamma^{-2} X_\infty \\ 0 & I \end{bmatrix}, \quad (24)$$

the stable invariant subspace of Riccati equation for Z is given by

$$T \begin{bmatrix} I \\ Y_\infty \end{bmatrix} = \begin{bmatrix} I - \gamma^{-2} X_\infty Y_\infty \\ Y_\infty \end{bmatrix}. \quad (25)$$

The stable solution of Riccati equation of Eq.(23) is $Y_\infty (I - \gamma^{-2} X_\infty Y_\infty)^{-1}$. We can know that $Z = Y_\infty (I - \gamma^{-2} X_\infty Y_\infty)^{-1} = (I - \gamma^{-2} Y_\infty X_\infty)^{-1} Y_\infty = Z_\infty Y_\infty$ from simple algebraic calculation. The stable error dynamics of riccati Eq.(23) is expressed by

$$\begin{aligned} \dot{e} = &(A + \gamma^{-2} B_1 B_1^T X_\infty + \gamma^{-2} Z_\infty Y_\infty X_\infty B_2 B_2^T X_\infty \\ &+ Z_\infty H_\infty C_2) e, \end{aligned} \quad (26)$$

and this is error dynamics which can be stabilized by observer and feedback gain. The worst-case state estimation can be found from the difference between the real error dynamics of Eq.(21) and error dynamics which can be stabilized like Eq.(26). Therefore, the worst-case state estimation is $\frac{1}{\gamma^2} Z_\infty Y_\infty X_\infty B_2 B_2^T X_\infty$: that is the stabilizable maximum perturbation for state error dynamics. As shown in Fig.2, the system matrix of the plant constrained by the infinity norm includes the bad-effect which can be caused by worst-case disturbance($\frac{1}{\gamma^2} B_1 B_1^T X_\infty$) and worst-case state estimation($\frac{1}{\gamma^2} Z_\infty Y_\infty X_\infty B_2 B_2^T X_\infty$). With the mathematical expression, the system matrix of plant constrained by the infinity norm like Eq.(16) includes implicitly Riccati solutions related with feedback and observer gain matrices dependent upon γ. If $\gamma \to \infty$, then we can obtain original plant of $C_2 (sI - A)^{-1} B_2$ from the plant constrained by infinity norm constraint of Eq.(16).

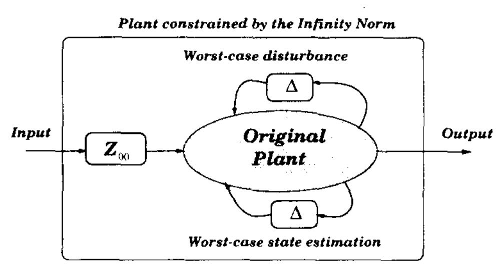

Fig. 2 Graphical interpretation of the plant constrained by the infinity norm

3.2. *Constrained Doubly Coprime Factorization*

In this section, we show the doubly coprime factorization for the plant constrained by the infinity norm. The lower linear fractional transformation of a generalized plant P on a controller K can be expressed as $\mathcal{F}_l(P, K)$, and $\|\mathcal{F}_l(P, K)\|_\infty \le \gamma$ is equal to the infinity norm constraint of Eq.(10). Let's the subscript "r" represent "constrained right coprime factor" and "l" "constrained left coprime factor".

Theorem 1 *Given a generalized plant satisfying five assumptions*

$$P = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right], \quad (27)$$

if X_∞, Y_∞ and Z_∞ exist, then we can find the plant constrained by infinity norm constraint of Eq.(16). Let's $P_s = N_r M_r^{-1} = M_l^{-1} N_l$ be right and left coprime factorization of P_s, and let

$$\begin{bmatrix} Y_r & X_r \\ -N_l & M_l \end{bmatrix} \begin{bmatrix} M_r & -X_l \\ N_r & Y_l \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \quad (28)$$

with each transfer matrices in $\mathcal{RH}_\infty$, and where

$$\begin{bmatrix} Y_r & X_r \\ -N_l & M_l \end{bmatrix} = \left[\begin{array}{c|cc} A_r & Z_\infty B_2 & -Z_\infty H_\infty \\ \hline -F_\infty & I & 0 \\ -C_2 & 0 & I \end{array}\right] \quad (29)$$

and

$$\begin{bmatrix} M_r & -X_l \\ N_r & Y_l \end{bmatrix} = \left[\begin{array}{c|cc} A_l & Z_\infty B_2 & -Z_\infty H_\infty \\ \hline F_\infty & I & 0 \\ C_2 & 0 & I \end{array}\right] \quad (30)$$

where $A_r = A + \frac{1}{\gamma^2} B_1 B_1^T X_\infty + \frac{1}{\gamma^2} Z_\infty Y_\infty X_\infty B_2 B_2^T X_\infty + Z_\infty H_\infty C_2$ and $A_l = A + \frac{1}{\gamma^2} B_1 B_1^T X_\infty + B_2 F_\infty$.

If X_∞, Y_∞ and Z_∞ exist, the left and right coprime factors constrained by the infinity norm can be found in any cases. These coprime factors of Eq.(29) and Eq.(30) are useful in that it can parameterize all stabilizing controllers which is constrained by the infinity norm, and easily make the sensitivity and closed-loop transfer matrix.

4. NUMERICAL EXAMPLE

Our example is to find controller K which can satisfy $\|\mathcal{F}_l(P, K)\|_\infty \leq \gamma$. Assume the generalized plant is given by

$$P = \left[\begin{array}{cc|c} G & I & G \\ 0 & 0 & I \\ \hline G & I & G \end{array}\right] \quad (31)$$

and if $G(s) = C(sI - A)^{-1} B$, then

$$P = \left[\begin{array}{c|cc} A & \begin{bmatrix} B & 0 \end{bmatrix} & B \\ \hline \begin{bmatrix} C \\ 0 \end{bmatrix} & \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 \\ I \end{bmatrix} \\ C & \begin{bmatrix} 0 & I \end{bmatrix} & 0 \end{array}\right]. \quad (32)$$

Since the generalized plant(P) does not satisfy five all assumptions, we must use the loop transformation and scaling($S_1^{-1} = I$ and $S_2^{-1} = \sqrt{1 - \gamma^{-2} I}$)).[1] Then, we can find the generalized plant satisfying five all assumptions. And the lowest achievable γ_{opt} is expressed by

$$\gamma_{opt} = \sqrt{1 + \rho(X_2 Y_2)}. \quad (33)$$

The plant constrained by infinity norm for the generalized plant satisfying five all assumptions is described as

$$P_s = \left[\begin{array}{c|c} A + \frac{1}{\gamma^2}(I + Z_\infty Y_\infty X_\infty) B B^T X_\infty & Z_\infty B \\ \hline \frac{1}{\sqrt{1-\gamma^{-2}}} C & 0 \end{array}\right] \quad (34)$$

and left and right coprime factors of Theorem 1 for P_s are

$$\begin{bmatrix} Y_r & X_r \\ -N_l & M_l \end{bmatrix} = \left[\begin{array}{c|cc} A_r & Z_\infty B & \frac{1}{\sqrt{1-\gamma^{-2}}} Z_\infty Y_\infty C^T \\ \hline B^T X_\infty & I & 0 \\ -\frac{1}{\sqrt{1-\gamma^{-2}}} C & 0 & I \end{array}\right] \quad (35)$$

and

$$\begin{bmatrix} M_r & -X_l \\ N_r & Y_l \end{bmatrix} = \left[\begin{array}{c|cc} A_l & Z_\infty B & \frac{1}{\sqrt{1-\gamma^{-2}}} Z_\infty Y_\infty C^T \\ \hline -B^T X_\infty & I & 0 \\ \frac{1}{\sqrt{1-\gamma^{-2}}} C & 0 & I \end{array}\right] \quad (36)$$

where $A_r = A + \frac{1}{\gamma^2}(I + Z_\infty Y_\infty X_\infty) B B^T X_\infty - \frac{1}{1-\gamma^{-2}} Z_\infty Y_\infty C^T C$ and $A_l = A - (1 - \gamma^{-2}) B B^T X_\infty$. The infinity norm of closed-loop transfer matrix can be expressed as follows, by letting $G = M_{ul}^{-1} N_{ul}$ and $K = S_1^{-1} X_l Y_l^{-1} S_2^{-1} (Q = 0)$. The closed-loop transfer matrix is expressed by

$$\|\mathcal{F}_l(P, K)\|_\infty = \left\| \begin{bmatrix} Y_l \\ -S_2^{-1} X_l \end{bmatrix} U^{-1} \right\|_\infty \leq \gamma \quad (37)$$

where $U = M_{ul} Y_l + S_2^{-1} N_{ul} X_l$.

Let $G(s) = \frac{1}{s^2+1}$, for the given plant, we know $\gamma_{opt} = 1.8$ by Eq.(33). For a $\gamma > \gamma_{opt}$, we can make constrained left and right coprime factors constrained by the infinity norm according to Eq.(35) and Eq.(36). If $\gamma = 1.9$,

$$\|\mathcal{F}_l(P, K)\|_\infty = 1.8924, \quad (38)$$

and if $\gamma = 1.81$,

$$\|\mathcal{F}_l(P, K)\|_\infty = 1.8099. \quad (39)$$

As shown above, we can easily satisfy the infinity norm constraint by using constrained coprime factors.

5. CONCLUDING REMARKS

Constrained left/right coprime facotrs has been suggested in which ∞-norm of closed-loop transfer matrix can be smaller than given constant γ without using optimization such as matrix dilation one. If the worst-case disturbance and worst-case state estimation are included in the original plant, the plant constrained by the infinity norm can be obtained. The closed-loop transfer matrix can easily be obtained by using unconstrained/constrained coprime factors.

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